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Interaction of Ionic Solution with Permeable Membranes: a Variational Approach

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Abstract. The movement of ionic solutions is an essential part of biology and technology. Fluidics, from nano-to microfluidics, is a burgeoning area of technology which is all about the movement of ionic solutions, on various scales. Many cells, tissues, and organs of animals and plants depend on osmosis, as the movement of fluids is called in biology. Indeed, the movement of fluids through channel proteins (that have a hole down their middle) is fluidics on an atomic scale. Ionic fluids are complex fluids, with energy stored in many ways. Ionic fluid flow is driven by gradients of concentration, chemical and electrical potential, and hydrostatic pressure. In this paper, a series of sharp interface models are derived for ionic solution with permeable membranes. By using the energy variation method, the unknown flux and interface conditions are derived consistently. We start from the derivation the generic model for the general case that the density of solution varies with ionic solvent concentrations and membrane is deformable. Then the constant density and fix membrane cases are derived as special cases of the generic model.

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1 Introduction

Osmosis moves ionic solutions throughout biology and technology. It is hard to find a more widespread phenomenon. All biology occurs in ionic solutions [1, 2] that move on many scales, including atomic scales smaller than nanometers, and a great deal of chemistry involves ionic movement as well. The modern technology of fluidics — macro, micro and nano — usually moves water and ions and thus involves osmosis at every scale. Indeed, the modern names "x-fluidics" (where x=nano, micro, and someday pico) may be viewed as a renaming of a classical [3], if not ancient word [4] into more modern language.

Ionic solutions involve energy stored in many forms, pressure, concentration, electric and electrochemical potential, steric interactions, and chemical energy (We define 'chemical' forces between atoms as those that significantly change the spatial distribution of electrons in the atoms. In this definition, dielectric interactions are not a chemical force. More precisely, force produced by the induced charge, that is proportional to the local electric field, is not a chemical force. Dispersion forces arise from the quantum fluctuations in induced ('dielectric') forces and so are not considered chemical in this definition). At the same time, ionic solutions always include flows of many types, usually in a coherent consistent way. Here, 'Consistent' means that all variables satisfy all equations and all boundary conditions with one set of parameters. Ions interact and move as components of complex fluids [5–12]. The solution itself flows more or less as water itself would move (without the ions). Water molecules and ions move (partly) by bulk flow, that is to say, they move (partly) by convection described classically by Navier Stokes equations. Water molecules and ions also move (partly) by diffusion. Ions move (partly) because of the electric field. Water also moves in an electric field because of dielectrophoresis [13,14] when the electric field is nonuniform i.e., $|\nabla \boldsymbol{E}| > 0$, where $\boldsymbol{E}$ is the electric field. Each of these flows varies with location and is described by field equations — typically partial differential equations in space and time — along with boundary conditions, that idealize the physics of the particular setup in which the flows occur. So ionic solutions are complex fluids [5, 15, 16].

The theory of complex fluids has developed consistent theories for many complex systems [10,17–19] that are mixtures (or even ionic solutions) with components that store energy. Variational methods deal successfully with magnetohydrodynamics systems [20], liquid crystals, polymeric fluids [11,12], colloids and suspensions [17,21] and electrorheological fluids [22]. The variational approach is widely used to analyze complex fluids because it derives field equations and boundary (or interface boundary) conditions, rather than assumes them. Once energy and dissipation functionals are defined by a physical model, the interaction terms in the field equations for each flow (bulk flow, diffusional