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Local Existence of Strong Solutions to the Generalized MHD Equations*

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Abstract This paper devotes to consider the local existence of the strong solutions to the generalized MHD system with fractional dissipative terms $\Lambda^{2\alpha}u$ for the velocity field and $\Lambda^{2\alpha}b$ for the magnetic field, respectively. We construct the approximate solutions by the Fourier truncation method, and use energy method to obtain the local existence of strong solutions in $H^s(\mathbb{R}^n)$ $(s > \max\left\{\frac{n}{2}+1-2\alpha, 0\right\})$ for any $\alpha \geq 0$.

Keywords Generalized MHD system, local existence, Fourier truncation

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1. Introduction

In this paper, we consider the Cauchy problem of the following $n-$dimensional $(n \geq 2)$ generalized MHD (GMHD) equations:

$$u_t + u\cdot\nabla u + \nabla\pi + \Lambda^{2\alpha}u - b\cdot\nabla b = 0,\ (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, \tag{1.1}$$

$$b_t + u\cdot\nabla b + \Lambda^{2\beta}b - b\cdot\nabla u = 0,\ (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, \tag{1.2}$$

$$\mathrm{div}u = 0, \mathrm{div}b = 0,\ (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, \tag{1.3}$$

$$(u,b)(x,0) = (u_0,b_0), \qquad x\in\mathbb{R}^n, \tag{1.4}$$

where $u = u(x,t)\in\mathbb{R}^n, b = b(x,t)\in\mathbb{R}^n$ and $\pi = \pi(x,t)\in\mathbb{R}$ denote the velocity field, magnetic field and scalar pressure respectively. $\alpha \geq 0$ is a real parameter. The fractional Laplacian operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ is defined through the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where the Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

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For the local existence results related to our equations, when $\alpha = 1,\ \beta = 0$, Fefferman et al. [1] established the local-in-time existence and uniqueness of strong solutions in $H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$ $(n = 2,\ 3)$. When $\alpha \geq 0,\ \beta > 0$, Wu [2] proved that the system has a unique local solution in $H^s(\mathbb{R}^n),\ s > max\{2, \beta\} + \frac{n}{2}$. When $\alpha = \beta \in (0,\ \frac{3}{2})$, Yuan [3] obtained the local existence of solution in $H^s(\mathbb{R}^3),\ s > \frac{5}{2} - 2\alpha$. For generalized $\alpha,\ \beta \geq 0$, Jiang and Zhou [4] proved the local existence results in $H^s(\mathbb{R}^n)$ with $s > max\{\frac{n}{2} + 1 - \alpha,\ 1\}$.

For the global existence results related to our equations, when $\alpha \geq \frac{1}{2} + \frac{n}{4},\ \alpha + \beta \geq 1 + \frac{n}{2}$, Zhou [5] established the global existence of solutions. When $\frac{1}{2} < \alpha,\ \beta \leq 1$, Ye [6] proved the global existence of mild solutions and small solutions in Fourier-Herz space. When $0 < \alpha < \frac{1}{2},\ \beta > 1,\ 3\alpha + 2\beta > 3$, Cheng [7] showed the existence of global regular solutions for logarithmically supercritical $2-$dimensional GMHD equations. Zhao [8] established the decay results for $\alpha,\ \beta \in (0, 2]$ when $u_0,\ b_0 \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ $(p > 1)$. Some regularity criteria were studied for the GMHD system in [9–11].

Now, we introduce some notations that will be used in our paper. $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^n)$ norm. $\|u\|_{H^s(\mathbb{R}^n)}$ and $\|u\|_{\dot{H}^s(\mathbb{R}^n)}$ denote the norm of u in the non-homogeneous Sobolev spaces $H^s(\mathbb{R}^n)$ and homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ respectively. C denotes a generic positive constant which may vary from line to line.

In this paper, we will consider the case $\alpha = \beta$ of GMHD system and establish the local existence of strong solutions. The main result of our paper is given by the following Theorem.

Theorem 1.1. *For any $\alpha \geq 0$, if $u_0,\ b_0 \in H^s(\mathbb{R}^n)$ with $s > \max\left\{\frac{n}{2} + 1 - 2\alpha,\ 0\right\}$, $\mathrm{div}u_0 = \mathrm{div}b_0 = 0$, then there exists a positive time T_* and a unique solution $(u,\ b)$ to equations* (1.1)-(1.4) *on* $[0,\ T_*]$ *such that*

$$u, b \in L^\infty([0, T_*];\ H^s(\mathbb{R}^n)) \cap L^2([0, T_*];\ H^{s+\alpha}(\mathbb{R}^n)).$$

Moreovre, we could obtain $u, b \in C_w([0, T_*];\ H^s(\mathbb{R}^n))$.

This paper is organized as follows. In Section 2, we introduce some preliminary lemmas that will be used frequently in the proof of our main results, and establish the energy estimates for the solutions of (1.1)-(1.4). In Section 3, we construct the approximate solutions, and give the proof of Theorem 1.1.

2. Preliminaries and energy estimates

In this section, we recall some elementary lemmas and give energy estimates for smooth solutions of (1.1)-(1.4), which are crucial in the proof of Theorem 1.1.

2.1. Preliminaries

Lemma 2.1. *[1] Define the Fourier truncation S_R as follows:*

$$\widehat{S_R f}(\xi) = 1_{B_R(\xi)}\hat{f}(\xi) = \begin{cases} \hat{f}(\xi), & |\xi| \leq R, \\ 0, & |\xi| > R, \end{cases}$$