Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 1, March 2024, 142–151 DOI:10.12150/jnma.2024.142

Existence and Uniqueness of Solutions for Time-Fractional Oldroyd-B Fluid Equations with Generalized Fractional Derivatives

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Abstract In this paper, we study the existence and uniqueness of solutions for time-fractional Oldroyd-B fluid equations with generalized fractional derivatives. We distinguish two cases. Firstly for the linear case, we get regularity results under some hypotheses of the source function and the initial data. Secondly for the nonlinear case, we use the Banach fixed point theorem to obtain the existence and uniqueness of solutions.

Keywords Time-fractional Oldroyd-B fluid equations, generalized fractional derivatives, generalized Laplace transform, regularity, Banach fixed point theorem

MSC(2010) 35R11, 35B65, 26A33.

1. Introduction

The subject of fractional calculus has gained considerable popularity and importance over the past three decades, primarity due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Indeed, it does provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. The Oldroyd-B model is a constitutive model used to describe the flow of viscoelastic fluids. This model can be regarded as an extension of the upper-convected Maxwell model and is equivalent to a fluid filled with elastic bead and spring dumbbells. The model is named after its creator Oldroyd [11]. Moreover, it is considered that the generalized fractional Oldroyd-B fluid model is a special case of non-Newtonian fluids that is critical in a wide range of industries and applied sciences. As a result, there are a lot of papers on this subject, with a lot of distinct research directions. Riemann-Liouville, Caputo, Hadamard, Riesz and other definitions for fractional derivatives and fractional integrals are now in use. We can refer the reader to some papers [1, 2, 5, 8–10, 12, 15, 16].

In [14], Tri considered the following initial problem for the time-fractional Ol-

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droyd -B fluid equation

$$\begin{cases} \left(1+a\partial_t^\alpha\right)u_t\left(x,t\right)=\mu\left(1+b\partial_t^\beta\right)\Delta u\left(x,t\right)+F\left(x,t,u\left(x,t\right)\right),\ x\in\mathcal{D},\ 0<t\le T,\\ u\left(x,t\right)=0,\ \ \left(x,t\right)\in\partial\mathcal{D}\times\left(0,T\right),\\ u\left(x,0\right)=u_0\left(x\right),\ I^{1-\alpha}u_t\left(x,0\right)=0,\ x\in\mathcal{D}, \end{cases} \tag{1.1}$$

where ∂_t^α is the Riemann-Liouville fractional derivative [17].

$$\partial_t^\alpha v\left(t\right):=\frac{\partial}{\partial t}\int_0^t\mu_{1-\alpha}\left(s\right)v\left(t-s,x\right)ds,\ \mu_\beta\left(s\right):=\frac{1}{\Gamma\left(\beta\right)}s^{\beta-1},\ \left(\beta>0\right). \tag{1.2}$$

Here u_0 is called the initial data and F is the source function. The author has studied the problem (1.1) for two cases. In the first case or the linear case, under some hypotheses of the source function and initial data, he obtained regularity results, and for the second case or the nonlinear case, he used Banach's fixed point theorem to prove the existence and uniqueness of the solution.

In [3], Al-Maskari et al. considered the following initial boundary-value problem for the time-fractional Oldroyd-B fluid equation

$$\left(1+a\partial_t^\alpha\right)u_t\left(x,t\right)=\mu\left(1+b\partial_t^\beta\right)\Delta u\left(x,t\right)+f\left(x,t\right),\ \text{in }\Omega\times\left(0,T\right]$$

with a homogeneous Dirichlet boundary condition

$$u\left(x,t\right)=0\text{ on }\partial\Omega\times\left(0,T\right],$$

and initial conditions

$$u\left(x,0\right)=v(x),\ \ \left(I^{1-\alpha}u_t\right)\left(x,0\right)=0\text{ in }\Omega,$$

where f and v are given functions, the parameters $\alpha,\beta\in(0,1)$, μ, a and b are positive constants, and ∂_t^α is the Riemann-Liouville fractional derivative given in (1.2), which established regularity results for the exact solution.

Motivated by the above works, in this paper we consider the following problem

$$\begin{cases} \left(1+a\partial_g^\alpha\right)u_t\left(x,t\right)=\mu\left(1+b\partial_g^\beta\right)\Delta u\left(x,t\right)+F\left(x,t,u\left(x,t\right)\right),\ x\in\mathcal{D},\ d<t\le T,\\ u\left(x,t\right)=0,\ \ \left(x,t\right)\in\partial\mathcal{D}\times\left(d,T\right),\\ u\left(x,d\right)=u_d\left(x\right),\ I_g^{1-\alpha}u_t\left(x,d\right)=0,\ x\in\mathcal{D}, \end{cases} \tag{1.3}$$

where $T>0$ is a fixed time, $0<\alpha<\beta<1$, $a,b,d\ge0$ and $\mu>0$ are given constant parameters, and ∂_g^α is the generalized fractional derivative given by

$$\left(\partial_g^\alpha f\right)(t)=\frac{\left(\frac{1}{g'(t)}\frac{d}{dt}\right)}{\Gamma\left(1-\alpha\right)}\int_d^t\left(g(t)-g(u)\right)^{-\alpha}f(u)g'(u)du, \tag{1.4}$$

with $g\in C^1\left(\left[d,T\right],\mathbb{R}\right)$ such that $g'(t)>0$ for any $t\in[d,T]$. It can be easily noticed that when $g(t)=t$, (1.4) is the classical Riemann-Liouville fractional derivative and when $g(t)=\ln t$, (1.4) is the Hadamarad fractional derivative [7,13], and $\left(I_g^\alpha f\right)(t)$ is the generalized fractional integral given by

$$\left(I_g^\alpha f\right)(t)=\frac{1}{\Gamma\left(\alpha\right)}\int_d^t\left(g(t)-g(u)\right)^{\alpha-1}f\left(u\right)g'\left(u\right)du. \tag{1.5}$$