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On Hodge Decomposition, Effective Viscous Flux and Compressible Navier-Stokes

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Abstract. It has been known, since the pioneering works by Serre, Hoff, Vaĭgant-Kazhikhov, Lions and Feireisl, among others, the regularizing properties of the effective viscous flux and its characterization as the function whose gradient is the gradient part in the Hodge decomposition of the Newtonian force of the fluid, when the shear viscosity of the fluid is constant. In this article, we explore further the connection between the Hodge decomposition of the Newtonian force and the regularizing properties of its gradient part, by addressing the problem of the global existence of weak solutions for compressible Navier-Stokes equations with both viscosities depending on a spatial mollification of the density.

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1 Introduction

The dynamics of a viscous compressible fluid are modeled by the well known Navier-Stokes equations. For a barotropic fluid, the Navier-Stokes equations may be written as

$$\partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0, \tag{1.1}$$

$$\partial_t(\rho\mathbf{u})+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla P=\operatorname{div}\mathbb{S}, \tag{1.2}$$

where ρ and $\mathbf{u}$ are the density and the velocity field of the fluid, respectively, $P=P(\rho)$ is the pressure, $\mathbb{S}$ is the viscous stress tensor. Note that we are neglecting possible external forces for simplicity.

Let $T>0$ be fixed. Throughout this paper we consider Eqs. (1.1)-(1.2) posed on a smooth bounded open set $\Omega\subseteq\mathbb{R}^N$ with $N\geq 2$, along with the following initial and boundary conditions:

$$\mathbf{u}(t,x)=0, \qquad 0<t<T, \quad x\in\partial\Omega, \tag{1.3}$$

$$\rho(0,x)=\rho_0(x), \quad \rho(0,x)\mathbf{u}(0,x)=m_0(x), \quad x\in\Omega. \tag{1.4}$$

Moreover, we assume that the fluid is isentropic and satisfies

$$P(\rho)=A\rho^\gamma \tag{1.5}$$

for some constants $A>0$ and $\gamma>N/2$. Let us consider the case of a Newtonian fluid, where the viscous stress tensor takes the form

$$\mathbb{S}=\lambda(\operatorname{div}\mathbf{u})\mathbb{I}+2\mu\mathbb{D}(\mathbf{u}), \tag{1.6}$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^N, \mathbb{D}(\mathbf{u})=(\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)/2$ is the symmetric part of the velocity gradient and λ and μ are the viscosity coefficients which, in general, depend on the density.

It has been known, since the pioneering works by Serre [24], Hoff [17, 18], Vaĭgant and Kazhikhov [26], Lions [21] and Feireisl [13, 14], among others (see also, e.g. [10, 20, 23, 25]), the regularizing properties of the effective viscous flux and its characterization as the function whose gradient is the gradient part in the Hodge decomposition of the Newtonian force of the fluid, when the shear viscosity of the fluid is constant.

The problem of existence of global solutions to the compressible Navier-Stokes equations with viscosity coefficients depending on the density is a difficult problem, specially in dimensions greater than one. The general theory developed by Lions [21], and later extended by Feireisl, Novotný and Petzeltová [16] and by