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The Relaxation Limit of a Quasi-Linear Hyperbolic-Parabolic Chemotaxis System Modeling Vasculogenesis

Qingqing Liu[1], Hongyun Peng[2] and Zhi-An Wang[3,*]

[1] *School of Mathematics, South China University of Technology, Guangzhou 510641, P.R. China.*
[2] *School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510006, P.R. China.*
[3] *Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong SAR, P.R. China.*

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Abstract. This paper is concerned with the relaxation limit of a three-dimensional quasi-linear hyperbolic-parabolic chemotaxis system modeling vasculogenesis when the initial data are prescribed around a constant ground state. When the relaxation time tends to zero (i.e. the damping is strong), we show that the strong-weak limit of the cell density and chemoattractant concentration satisfies a parabolic-elliptic Keller-Segel type chemotaxis system in the sense of distribution.

AMS subject classifications: 35L60, 35L04, 35B40, 35Q92

Key words: Hyperbolic-parabolic model, vasculogenesis, diffusion, relaxation limit.

1 Introduction

In this paper, we are concerned with the following quasi-linear hyperbolic-parabolic chemotaxis system modeling vasculogenesis – the vitro formation of new blood vessels, proposed by Gamba *et al.* [11]:

*Corresponding author. *Email addresses:* maqqliu@scut.edu.cn (Q. Liu), penghy010@163.com (H. Peng), mawza@polyu.edu.hk (Z.-A. Wang)

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, & \text{(1.1a)} \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) = -\dfrac{1}{\tau} \rho u + \mu \rho \nabla \phi, & \text{(1.1b)} \\ \partial_t \phi = D \Delta \phi + a\rho - b\phi. & \text{(1.1c)} \end{cases}$$

Here the unknowns $\rho = \rho(x,t)$ and $u = u(x,t) \in \mathbb{R}^3$ denote the density and velocity of the endothelial cell, respectively, and $\phi = \phi(x,t)$ the chemoattractant concentration, at $t > 0$ and $x \in \mathbb{R}^3$. The density-dependent quantity P is the pressure function which is smooth and satisfies $P'(\rho) > 0$ for $\rho > 0$. D, a and b are positive constants representing the diffusion coefficient, production rate and degradation rate of the chemoattractant, $|\mu|$ with $\mu \in \mathbb{R} \setminus \{0\}$ is the cell response intensity to the chemoattractant. $0 < \tau \ll 1$ is a relaxation time. The initial data are given by

$$[\rho, u, \phi]|_{t=0} = [\rho_0, u_0, \phi_0](x) \;\rightarrow\; (\bar{\rho}, 0, \bar{\phi}) \quad \text{as} \quad |x| \;\rightarrow\; \infty \tag{1.2}$$

with constants $\bar{\rho} > 0$ and $\bar{\phi} > 0$. When the initial value $[\rho_0, u_0, \phi_0] \in H^s(\mathbb{R}^d)$, $s > d/2 + 1$, is a small perturbation of the constant ground state (i.e. equilibrium) $[\bar{\rho}, 0, \bar{\phi}]$ with $\bar{\rho} > 0$ sufficiently small, the global existence and stability of solutions to (1.1) without vacuum converging to $[\bar{\rho}, 0, \bar{\phi}]$ was established in [7,8]. By adding a viscous term Δu to the Eq. (1.1b), the linear stability of the constant ground state $[\bar{\rho}, 0, \bar{\phi}]$ was obtained in [17] under the condition

$$bP'(\bar{\rho}) - a\mu\bar{\rho} > 0. \tag{1.3}$$

The stationary solutions of (1.1) with vacuum (bump solutions) in a bounded interval with zero-flux boundary condition were constructed in [1,2]. Recently the stability of transition layer solutions of (1.1) on $\mathbb{R}_+ = [0, \infty)$ was established in [13] and the convergence to diffusion waves for solutions of (1.1) was obtained in [20] for $x \in \mathbb{R}^3$.

As $\tau \to 0$ (strong damping), it was formally derived in [4] by the asymptotic analysis that the solution of (1.1) converges to the well-known Keller-Segel model. In [6], the authors considered different dissipation relaxation limits of model (1.1), and proved in L^p $(p \geq 1)$ space that the convergence limit is the parabolic-elliptic Keller-Segel model by the energy methods and compensated compactness tools. An interesting question is whether the relaxation limit problem of (1.1) can be proved in a stronger sense, such as in H^s $(s \geq 3)$ space. For the isothermal compressible Euler equations, namely $P(\rho) = k\rho$ for some constant $k > 0$, by using a stream function, Junca and Rascle [15] showed that the solutions to the damped isothermal Euler equations converge to those of the heat equation for large BV initial data. Later, Coulombel and Goudon [5] studied the global existence of smooth solutions and the convergence to the heat equation as