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Multidomain Hybrid Direct DG and Central Difference Methods for Viscous Terms in Hyperbolic-Parabolic Equations

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Abstract. A class of multidomain hybrid methods of direct discontinuous Galerkin (DDG) methods and central difference (CD) schemes for the viscous terms is proposed in this paper. Both conservative and nonconservative coupling modes are discussed. To treat the shock wave, the nonconservative coupling mode automatically switch to conservative coupling mode to preserve the conservative property when discontinuities pass through the artificial interface. To maintain the accuracy of the hybrid methods, the Lagrange interpolation polynomials and their derivatives are reconstructed to handle the coupling cells in the DDG subdomain, while the values of ghost points for the CD subdomain are calculated by the approximate polynomials from the DDG methods. The linear stabilities of these methods are demonstrated in detail through von-Neumann analysis. The multidomain hybrid DDG and CD methods are then extended to one- and two-dimensional hyperbolic-parabolic equations. Numerical results validate that the multidomain hybrid methods are high-order accurate in the smooth regions, robust for viscous shock simulations and capable to save computational cost.

AMS subject classifications: 65M60, 65M06, 65M99, 35L65, 35K20

Key words: Direct discontinuous Galerkin, central difference schemes, multidomain hybrid methods, viscous terms, hyperbolic-parabolic equations.

1. Introduction

To accurately predict the engineering problems and fundamental flow physics, high-order methods, such as discontinuous Galerkin methods (DG) [9–13, 26] and weighted essentially non-oscillatory schemes (WENO) [19, 20, 31, 33, 34, 38, 48], have attracted more interest recently with the lower numerical dissipation. Except for those methods,

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high-order hybrid methods have been developed to combine the advantages of each individual method.

Luo *et al.* [24] presented a reconstruction-based discontinuous Galerkin (RDG (P1P2)) method, where the quadratic polynomial solution (P2) is obtained from the underlying linear polynomial (P1) discontinuous Galerkin solution using a least-squares method. Zhang *et al.* [42–45] proposed a hybrid DG/FV scheme, where the lower-order derivatives of a piecewise polynomial solution are computed locally in a cell by the DG methods and the higher-order derivatives are reconstructed by the known lower-order derivatives. Later, Zhao *et al.* [46] optimized the reconstruction strategy for DG/FV methods by a hierarchical reconstruction strategy, where the cell average and its derivatives were reconstructed by WENO reconstruction. Zhu *et al.* [47] and Guo *et al.* [16] developed a hybrid WCNS-CPR scheme for the efficient supersonic simulations, where WCNS is adopted to capture shocks while the smooth area is calculated by CPR. Maltsev *et al.* [25] developed a hybrid DG/FV schemes, in which the key ingredient is a switch between DG method and FV method based on the CWENOZ scheme. Based on the computational domain decomposition, Cheng *et al.* [4, 6, 7] proposed the multidomain hybrid RKDG and WENO methods for solving the hyperbolic conservation laws, which combined the advantages of high efficiency of the WENO schemes and easy treatment of the complex geometries easily from the DG methods. Later, Zhang *et al.* [41] analyzed the linear stabilities of the conservative multidomain hybrid methods and introduced two ways of healing the stable problems. Moreover, Wang *et al.* [37] proposed a novel high-order FD scheme based on DG boundary treatment and no more than two layers were needed for the complex boundary treatments. Up to date, the multidomain hybrid methods have been proposed to treat the inviscid terms in the compressible inviscid flow problems. To take the advantages of the method in solving compressible viscous flow problems, we extend the multidomain hybrid methods to handle the viscous terms and then apply the methods to solve hyperbolic-parabolic equations.

Central difference schemes are often used to discretize the viscous terms in the finite difference methods. Compared to the formulas used in [15, 49], Shen *et al.* [27–30] proposed a set of conservative fourth- and sixth-order central difference schemes for compressible flows with variable viscosity coefficient, which has the stencil width matching that of the fifth- and seventh-order WENO schemes and maintains the compactness of the WENO schemes. It is conservative and highly efficient but difficult to handle the complex geometries just as finite difference WENO schemes. In the DG methods, taking a simple arithmetic mean of the solution derivatives from the left and right is inconsistent [32]. A number of numerical methods have been proposed in the literature to address this issue. Among those methods, the direct discontinuous Galerkin methods [22, 23] are based on the direct weak formulation for solving the parabolic equations. The viscous numerical flux constructed in the DDG methods is consistent and conservative and no auxiliary variable required during the calculation. Later, Cheng *et al.* [5, 8, 40] extended the DDG methods to discretize the viscous and heat fluxes in the Navier-Stokes equations. The DDG methods are simple, conserva-