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On the Monotonicity of Q^2 Spectral Element Method for Laplacian on Quasi-Uniform Rectangular Meshes

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Abstract. The monotonicity of discrete Laplacian implies discrete maximum principle, which in general does not hold for high order schemes. The Q^2 spectral element method has been proven monotone on a uniform rectangular mesh. In this paper we prove the monotonicity of the Q^2 spectral element method on quasi-uniform rectangular meshes under certain mesh constraints. In particular, we propose a relaxed Lorenz's condition for proving monotonicity.

AMS subject classifications: 65N30, 65N06, 65N12

Key words: Inverse positivity, discrete maximum principle, high order accuracy, monotonicity, discrete Laplacian, quasi uniform meshes, spectral element method.

1 Introduction

In many applications, monotone discrete Laplacian operators are desired and useful for ensuring stability such as discrete maximum principle or positivity-preserving of physically positive quantities [6, 10, 18, 21]. Let Δ_h denote the matrix representation of a discrete Laplacian operator, then it is called *monotone* if $(-\Delta_h)^{-1} \geq 0$, i.e., the inverse matrix $(-\Delta_h)^{-1}$ has nonnegative entries. In this paper, all inequalities for matrices are entrywise inequalities.

In the literature, the most important tool for proving monotonicity is via nonsingular M-matrices, which are inverse-positive matrices. See the Appendix for a convenient characterization of the M-matrices. The simplest second order accurate centered finite difference $u''(x_i) \approx \frac{u(x_{i-1})-2u(x_i)+u(x_{i+1})}{\Delta x^2}$ is monotone because the corresponding matrix $(-\Delta_h)^{-1}$ is an M-matrix thus inverse positive. Even though the linear finite element method forms an M-matrix on unstructured triangular meshes under a mild mesh constraint [24], in

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general the discrete maximum principle is not true for high order finite element methods on unstructured meshes [9]. On the other hand, there exist a few high order accurate inverse positive schemes on structured meshes.

For solving a Poisson equation, provably monotone high order accurate schemes on structured meshes include the classical 9-point scheme [3, 7, 11] in which the stiffness matrix is an M-matrix. The classical 9-point scheme has the same stiffness matrix as fourth order accurate compact finite difference schemes [13], see the appendix in [16]. In [2,4], a fourth order accurate finite difference scheme was constructed and its stiffness matrix is a product of two M-matrices thus monotone. The Lagrangian P^2 finite element method on a regular triangular mesh [23] has a monotone stiffness matrix [19]. On an equilateral triangular mesh, the discrete maximum principle of P^2 element can also be proven [9]. Monotonicity was also proven for the Q^2 spectral element method on an uniform rectangular mesh for a variable coefficient Poisson equation under suitable mesh constraints [14]. The Q^k spectral element method is the continuous finite element method with Lagrangian Q^k basis implemented by $(k+1)$-point Gauss-Lobatto quadrature. The monotonicity of Q^3 spectral element method for Laplacian on uniform meshes was also proven in [8].

For proving inverse positivity, the main viable tool in the literature is to use M-matrices which are inverse positive. A convenient sufficient condition for verifying the M-matrix structure is to require that off-diagonal entries must be non-positive. Except the fourth order compact finite difference, all high order accurate schemes induce positive off-diagonal entries, destroying M-matrix structure, which is a major challenge of proving monotonicity. In [2] and [1], and also the appendix in [14], M-matrix factorizations of the form $(-\Delta_h)^{-1} = M_1 M_2$ were shown for special high order schemes but these M-matrix factorizations seem ad hoc and do not apply to other schemes or other equations. In [19], Lorenz proposed some matrix entry-wise inequality for ensuring a matrix to be a product of two M-matrices and applied it to P^2 finite element method on uniform regular triangular meshes.

In [14], Lorenz's condition was applied to Q^2 spectral element method on uniform rectangular meshes. Such a monotonicity result implies that the Q^2 spectral element method is bound-preserving or positivity-preserving for convection diffusion equations including the Allen-Cahn equation [21], the Keller-Segel equation [10], the Fokker-Planck equation [17], as well as the internal energy equation in compressible Navier-Stokes system [18]. On the other hand, all these results about Q^2 spectral element method are on uniform meshes. For both theoretical and practical interests, a natural question to ask is whether such a monotonicity result still holds on non-uniform meshes. The monotonicity of high order schemes on quasi-uniform meshes are preferred in many applications, e.g., [22].

The focus of this paper is to discuss Lorenz's condition for Q^2 spectral element method on quasi-uniform meshes. We discuss and derive sufficient mesh constraints to preserve monotonicity of Q^2 spectral element method on a quasi-uniform rectangular mesh. In general, the same discussion also applies to Lagrangian P^2 finite element method on a