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Boundedness of Commutators for Multilinear Marcinkiewicz Integrals with Generalized Campanato Functions on Generalized Morrey Spaces

Fuli Ku*

School of Information Engineering, Sanming University, Sanming 365004, China.

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Abstract. This paper is devoted to exploring the mapping properties for the commutator $\mu_{\Omega,\vec{b}}$ generated by multilinear Marcinkiewicz integral operators μ_Ω with a locally integrable function $\vec{b}=(b_1,\cdots,b_m)$ on the generalized Morrey spaces. $\mu_{\Omega,\vec{b}}$ is bounded from $L^{(p_1,\varphi_1)}(\mathbb{R}^n)\times\cdots\times L^{(p_m,\varphi_m)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, where $L^{(p_i,\varphi_i)}(\mathbb{R}^n),L^{(q,\varphi)}(\mathbb{R}^n)$ are generalized Morrey spaces with certain variable growth condition, that $b_j(j=1,\cdots,m)$ is a function in generalized Campanato spaces, which contain the $\mathrm{BMO}(\mathbb{R}^n)$ and the Lipschitz spaces $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ $(0<\alpha\le 1)$ as special examples.

AMS subject classifications: 42B20, 42B25

Key words: Multilinear, Marcinkiewicz integrals, commutators, generalized Campanato spaces, generalized Morrey spaces.

1 Introduction

Let $\mathbb{R}^n$, $n\ge 2$, be the n-dimensional Euclidean spaces and S^{n-1} the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma=d\sigma(\cdot)$. Let Ω be a homogeneous function of degree zero on $(\mathbb{R}^n)^m$ satisfying integration of Ω on $(B(0,1))^m$ vanishes,

$$\int_{(B(0,1))^m}\frac{\Omega(y)}{|y|^{m(n-1)}}dy=0. \tag{1.1}$$

The multilinear Marcinkiewicz integral operator μ_Ω is defined by

$$\mu_\Omega(\vec{f})(x)=\Big(\int_0^\infty|F_{\Omega,t}(\vec{f})(x)|^2\frac{dt}{t}\Big)^{\frac{1}{2}},$$

*Corresponding author. *Email address:* kfl20180325@163.com (Ku F)

where

$$F_{\Omega,t}(\vec{f})(x)=\frac{1}{t^m}\int_{(B(x,t))^m}\frac{\Omega(x-y_1,\cdots,x-y_m)}{(\sum_{i=1}^m|x-y_i|)^{m(n-1)}}\prod_{i=1}^m f_i(y_i)dy_i.$$

When $m=1$, μ_Ω is the classical Marcinkiewicz integral, which belongs to the broad class of the Littlewood-Paley g-functions and plays important roles in harmonic analysis and partial differential equations. The research on the mapping properties of Marcinkiewicz integrals and its commutators in various function spaces has been an active topic. In 1958, Stein [1] first introduced the operator μ_Ω, which is the higher dimensional generalization of Marcinkiewicz integrals in one-dimension, and showed that μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1<p\le 2$ and weak type (1,1), provided $\Omega\in \mathrm{Lip}_\alpha(S^{n-1})$, $0<\alpha\le 1$. Subsequently, the boundedness of μ_Ω was studied extensively, see [2–6], etc. Moreover, the boundedness of commutators generated by μ_Ω with a locally integrable function has been paid numerous attentions, see [7–10], etc and therein references.

For $m\ge 2$, if Ω satisfies Lipschitz continuous condition on $(S^{n-1})^m$, i.e., there exist $0<\gamma<1$ and $C>0$ such that for any $\xi=(\xi_1,\dots,\xi_m)$, $\eta=(\eta_1,\dots,\eta_m)\in(\mathbb{R}^n)^m$,

$$|\Omega(\xi)-\Omega(\eta)|\le C|\xi'-\eta'|^\gamma,$$

where

$$y'=(y_1,\dots,y_m)'=\frac{(y_1,\dots,y_m)}{|y_1|+\dots+|y_m|}.$$

Under the above conditions, Chen-Xue-Yabuta [11] obtain μ_Ω strong $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)\to L^p(\nu_{\vec{\omega}})$ estimates when $p_i>1$ and weak type $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)\to L^{p,\infty}(\nu_{\vec{\omega}})$ estimates if there is a $p_i=1$.

In this paper, we will focus on the commutator $\mu_{\Omega,\vec{b}}$ generated by multilinear Marcinkiewicz integral operators μ_Ω with $b_j\in L^1_{\mathrm{loc}}(\mathbb{R}^n), j=1,\cdots,m$ by

$$\mu_{\Omega,\vec{b}}(\vec{f})(x):=\sum_{j=1}^m\mu_{\Omega,b_j}(\vec{f})(x)=\sum_{j=1}^m\Big(\int_0^\infty|[b_j,F_{\Omega,t}](\vec{f})(x)|^2\frac{dt}{t}\Big)^{\frac12},$$

where

$$[b_j,F_{\Omega,t}](\vec{f})(x)=b_j(x)F_{\Omega,t}(\vec{f})(x)-F_{\Omega,t}(f_1,\cdots,b_jf_j,\cdots,f_m)(x),$$
$$\mu_{\Omega,b_j}(\vec{f})(x):=\Big(\int_0^\infty\Big|\int_{(B(x,t))^m}\widetilde{\Omega}(x,\vec{y})(b_j(x)-b_j(y_j))\prod_{i=1}^m f_i(y_i)dy_i\Big|^2\frac{dt}{t^{2m+1}}\Big)^{\frac12},$$

where

$$\widetilde{\Omega}(x,\vec{y})=\frac{\Omega(x-y_1,\cdots,x-y_m)}{(\sum_{i=1}^m|x-y_i|)^{m(n-1)}}.$$

Recently, He and Liang [12] proved that $b_j\in\mathrm{BMO}(\mathbb{R}^n), j=1,\cdots,m$, then $\mu_{\Omega,\vec{b}}$ is bounded from $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)\to L^p(\nu_{\vec{\omega}})$ and from $\big(\prod_i^m L\log L(\omega_i)\big)^{1/m}$ to $L^{1,\infty}(\nu_{\vec{\omega}})$.