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Convergence Towards the Population Cross-Diffusion System from Stochastic Many-Particle System

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Abstract. In this paper, we derive rigorously a non-local cross-diffusion system from an interacting stochastic many-particle system in the whole space. The convergence is proved in the sense of probability by introducing an intermediate particle system with a mollified interaction potential, where the mollification is of algebraic scaling. The main idea of the proof is to study the time evolution of a stopped process and obtain a Grönwall type estimate by using Taylor's expansion around the limiting stochastic process.

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1 Introduction

In this paper, we give a rigorous justification for the mean-field limit from an interacting particle system to the population cross-diffusion system as the number of particles goes to infinity. More precisely, we present the derivation of n-species

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cross-diffusion system as follows:

$$\begin{cases} \partial_t u_i = \operatorname{div}(u_i \nabla U_i) + \sigma_i \Delta u_i + \operatorname{div}\left[u_i \sum_{j=1}^{n} \nabla f(B_{ij} * u_j) \right], \\ B_{ij}(|x|) = \dfrac{C(d,\vartheta_{ij})}{|x|^{\vartheta_{ij}}}, \quad \vartheta_{ij} \in (0, d-2], \\ u_i(0) = u_i^0(x), \qquad\qquad i = 1,\dots,n, \end{cases} \tag{1.1}$$

where $\sigma_i > 0$ are the constant diffusion coefficients, $\mathbf{u} = (u_1,\dots,u_n)$ stands for the vector of population densities, $U_i(x) = -|x|^2/2$ represent environment potentials and $C(d,\vartheta_{ij})$ are constants depend on d and ϑ_{ij}. The transitions rates depend on the densities by a nonlinear term f.

The aim of this paper is to rigorously derive the system (1.1) from the following stochastic many-particle system. This system describes the movements of n species of particles, with the particle numbers $N_i \in \mathbb{N}$, $i = 1,\dots,n$, according to the given law. Without loss of generality, we let $N = N_i$, $i = 1,\dots,n$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_{t\geq 0}), \mathbb{P})$ be a complete filtered probability space. We consider d-dimensional $\mathcal{F}_t$-Brownian motions $(W_i^k(t))_{t\geq 0}$, $k = 1,\dots,N$, $i = 1,\dots,n$ which are assumed to be independent of each other. We assume that (ξ_i^k), $k = 1,\dots,N$, $i = 1,\dots,n$ are i.i.d. random variables, independent of $(W_i^k(t))_{t\geq 0}$, and have common probability density function u_i^0. We use the notation $X_{\eta,i}^{N,k}(t)$ to represent the k-th particle of i-th species and the dynamics of $X_{\eta,i}^{N,k}(t)$ are governed by

$$\begin{cases} dX_{\eta,i}^{N,k} = \left[-\nabla U_i\left(X_{\eta,i}^{N,k}\right) - \sum_{j=1}^{n} \nabla f_\gamma \left(\dfrac{1}{N} \sum_{l=1}^{N} B_{ij}^{\eta}\left(X_{\eta,i}^{N,k} - X_{\eta,j}^{N,l} \right) \right) \right] dt \\ \qquad\qquad + \sqrt{2\sigma_i}\, dW_i^k(t), \\ X_{\eta,i}^{N,k}(0) = \xi_i^k, \quad i = 1,\dots,n, \quad k = 1,\dots,N, \end{cases} \tag{1.2}$$

where f_γ is an approximation of f which can be constructed, for example in Remark 1.1, and

$$B_{ij}^{\eta} := \begin{cases} V^\eta * B_{ij}, & 0 < \vartheta_{i,j} < d-2 \\ V^\eta * \bar{B}_{ij}, & \vartheta_{i,j} = d-2 \end{cases}, \qquad \bar{B}_{ij}(|x|) := \begin{cases} B_{ij}(|x|), & |x| \geq \eta \\ B_{ij}(\eta), & |x| < \eta \end{cases}.$$

Here $V^\eta(x) := V(x/\eta)/\eta^d$ with $\eta > 0$ is a mollification kernel which means $V \geq 0$ is a given radially symmetric smooth function such that $\int_{\mathbb{R}^d} V(x)dx = 1$.