JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **36**, No. 4, pp. 404-413

doi: 10.4208/jpde.v36.n4.6
December 2023

# The Lifespan of Smooth Solutions to Semilinear Wave Equations in Schwarzschild Space-Time

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Received 21 August 2021; Accepted 20 January 2023

**Abstract.** This paper considers the Cauchy problem of the semilinear wave equations with small initial data in the Schwarzschild space-time, $\Box_g u = |u_t|^p$, where $g$ denotes the Schwarzschild metric. When $1<p<2$ and the initial data are supported far away from the black hole, we can prove that the lifespan of the spherically symmetric solution obtains the same order as the semilinear wave equation evolving in the Minkowski space-time by introducing an auxiliary function.

**AMS Subject Classifications**: 35L05, 35L15

**Chinese Library Classifications**: O175.27

**Key Words**: Semilinear wave equations; Schwarzschild spacetime; blow-up; lifespan.

## 1 Introduction

In this paper, we consider the blow-up phenomenon of the solution to the following nonlinear wave equation

$$\Box_g u = |u_t|^p, \tag{1.1}$$

where $g$ denotes the metric of the Schwarzschild space-time and $p>1$.

In the Minkowski spacetime, the critical exponent is $p(n)=\frac{n+1}{n-1}$, in which $n$ is the spatial dimension. It is well-known that when $1<p\le p(n)$, the solution blows up even for small initial data; when $p>p(n)$, global solution for small initial data exists. The blow-up results were first established by F. John [1] when $n=3$. When $n=2$, such results were given by J. Schaeffer [2] as well as R. Agemi [3]. When $n=1$, K. Masuda [4] proved the same result. The radially symmetric solutions also blow up when $n\ge 4$ with $p=p(n)$ if $n$ is

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odd, and $1<p<p(n)$ when $n$ is even. The lifespan of solution was also discussed in some papers. Li and Chen [5] studied the lower bound of the lifespan. The lifespan is proved to be sharp as in Lax [6], F. John [7], Kong [8] and Y. Zhou [9], while the general result was obtained in Y. Zhou [10]. For the blow-up of solutions with variable coefficients on exterior domain, see Y. Zhou and W. Han in [11].

Recently, the study of hyperbolic partial differential equations in curved space-time has draw much attention of the mathematicians, due to the great development of the general relativity. We want to know whether the results of hyperbolic PDEs in Minkowski space-time are still hold in curved space-time. One can see [12–16] for the perfect fluids in FLRW spacetimes, and see [17–19] for nonlinear wave equations in de Sitter space-time. In this paper, we are interested in the lifespan of smooth solutions to the semilinear wave equations evolving in Schwarzschild space-time. When the nonlinear term is $|u|^p$ with $3/2\le p\le 2$, the lifespan has been studied by Lin, Lai and Ming [20]. When the nonlinear term is $|u_t|^p$ with $1<p<2$, Lai and Zhou [21] studied the lifespan of the spherically symmetric solution. The general results on the Glassey conjecture for all spatial dimensions with radially symmetric data is studied by Hidano Wang and Yokoyama, see [22]. Inspired by Y. Zhou and W. Han in [11], We consider the semilinear wave equations evolving in the Schwarzschild space-time when $1<p<p(n)$. We can prove that the lifespan is $T(\varepsilon)\le\tilde{C}\varepsilon^{-\frac{p-1}{2-p}}$, which has the same order as the Minkowski case.

### 1.1 Main theorem

Consider the nonlinear wave equations in the Schwarzschild spacetime $\mathcal{M}$

$$\Box_g u=|u_t|^p,\qquad p>1. \tag{1.2}$$

The Schwarzschild metric $g$ is

$$g=\left(1-\frac{2M}{r}\right)\mathrm{d}t^2-\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2-r^2\mathrm{d}\omega^2, \tag{1.3}$$

here $M>0$ denotes the mass of the universe and $d\omega^2$ is the standard metric on the unit sphere $\mathbf{S}^2$, $r>2M$. Denoting

$$F(r)=1-\frac{2M}{r}, \tag{1.4}$$

the D'Alembert operator associated with metric $g$ becomes

$$\Box_g=\frac{1}{F}\left(\partial_t^2-\frac{F}{r^2}\partial_r(r^2F\partial_r)-\frac{F}{r^2}\triangle_{\mathbf{S}^2}\right), \tag{1.5}$$

where $\triangle_{\mathbf{S}^2}$ is the standard Laplace-Beltrami operator on $\mathbf{S}^2$.

Define the Regge-Wheeler coordinate

$$s(r)=r+2M\log(r-2M) \tag{1.6}$$