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Second-Order Difference Equation for Sobolev-Type Orthogonal Polynomials. Part II: Computational Tools

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Abstract. We consider polynomials orthogonal with respect to a nonstandard inner product. In fact, we deal with Sobolev-type orthogonal polynomials in the broad sense of the expression. This means that the inner product under consideration involves the Hahn difference operator, thus including the difference operators $\mathscr{D}_q$ and Δ and, as a limit case, the derivative operator. In a previous work, we studied properties of these polynomials from a theoretical point of view. There, we obtained a second-order differential/difference equation satisfied by these polynomials. The aim of this paper is to present an algorithm and a symbolic computer program that provides us with the coefficients of the second-order differential/difference equation in this general context. To illustrate both, the algorithm and the program, we will show three examples related to different operators.

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Key words: Sobolev orthogonal polynomials, second-order difference equation, symbolic computation.

1. Introduction

In this paper we tackle the problem of computing symbolically the coefficients of the second-order differential/difference equation satisfied by the monic orthogonal polynomials $Q_n(x)$ with respect to the general discrete Sobolev-type inner product

$$(f,g)_S = \int f(x)g(x)\varrho(x)dx + M\mathscr{D}_{q,\omega}^{(j)}f(c)\mathscr{D}_{q,\omega}^{(j)}g(c), \tag{1.1}$$

where $\varrho(x)$ is a weight function on the real line, c is located on the real axis, $M>0$, j is a nonnegative integer, and $\mathscr{D}_{q,\omega}$ is the operator introduced by Hahn [5, Eq. (1.3)] defined

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by

$$\mathscr{D}_{q,\omega}f(x)=\begin{cases}\dfrac{f(qx+\omega)-f(x)}{(q-1)x+\omega}, & \text{if} \quad x\neq\omega_0,\\ f'(\omega_0), & \text{if} \quad x=\omega_0,\end{cases} \tag{1.2}$$

where $0<q<1$, $\omega\geq 0$, and $\omega_0=\omega/(1-q)$, cf. [8, Eq. (2.1.1)]. Besides, following [1], we define

$$\mathscr{D}^{(0)}_{q,\omega}f:=f, \quad \mathscr{D}^{(j)}_{q,\omega}f:=\mathscr{D}_{q,\omega}\mathscr{D}^{(j-1)}_{q,\omega}f, \quad j\geq 1.$$

It is well known that this class of operators includes the q-difference operator $\mathscr{D}_q$ by Jackson when $\omega=0$, the forward difference operator Δ when $q=1$ and $\omega=1$, and the derivative operator d/dx as a limit case when $\omega=0$ and $q\to 1$.

Theoretical backgrounds of this problem are established by [4, Theorems 4.1-4.2]. In particular, it was shown that the nonstandard orthogonal polynomials $Q_n(x)$ satisfy the second-order difference equation

$$\sigma_{1,c,n}(x)\mathscr{D}^{(2)}_{q,\omega}Q_n(x)+\sigma_{2,c,n}(x)\mathscr{D}_{q,\omega}Q_n(x)+\sigma_{3,c,n}(x)Q_n(x)=0, \quad n\geq 2, \tag{1.3}$$

where $\sigma_{1,c,n}(x)$, $\sigma_{2,c,n}(x)$, and $\sigma_{3,c,n}(x)$ are explicitly known functions. Moreover, there exist two operators Φ_n and $\widehat{\Phi}_n$, known as ladder operators, involving the operator of Hahn defined by (1.2) such that

$$\Phi_nQ_n(x)=\varphi^{1,2}_{c,n}(x)Q_{n-1}(x), \tag{1.4}$$

$$\widehat{\Phi}_nQ_{n-1}(x)=\varphi^{3,4}_{c,n}(x)Q_n(x), \tag{1.5}$$

where $\varphi^{1,2}_{c,n}(x)$ and $\varphi^{3,4}_{c,n}(x)$ can also be computed explicitly.

The study of second-order differential/difference equations and their solutions appear in several theoretical and applied contexts. Thus it is worth studying how to compute explicitly the polynomial coefficients $\sigma_{1,c,n}(x)$, $\sigma_{2,c,n}(x)$ and $\sigma_{3,c,n}(x)$ of (1.3). In this paper we present an algorithm highlighting its more important steps. Later, the symbolic program will be built using the programming language *Mathematica*® 13.1.0.[†] The corresponding code is freely available at `https://w3.ual.es/GruposInv/Tapo/SODE.nb`

The literature related to symbolic programs in the framework of Sobolev orthogonality is very recent. As far as we know, the first paper is [10], where Mehler-Heine formulas are computed symbolically — cf. `https://notebookarchive.org/2022-06-amlp3fh`, Notebook Archive (2022).

This paper is structured as follows. Section 2 is devoted to introducing theoretical results obtained in [4] as well as an algorithm to obtain symbolically the coefficients of the second-order differential/difference equation (1.3). In Section 3, we show how the program works for three examples related to different operators.

[†]The program in previous versions of *Mathematica*® may not work properly — e.g. we found malfunctions in the version 13.0.0.