A Normalizing Field Flow Induced Two-Stage Stochastic Homogenization Method for Random Composite Materials

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Abstract. The traditional stochastic homogenization method can obtain homogenized solutions of elliptic problems with stationary random coefficients. However, many random composite materials in scientific and engineering computing do not satisfy the stationary assumption. To overcome the difficulty, we propose a normalizing field flow induced two-stage stochastic homogenization method to efficiently solve the random elliptic problem with non-stationary coefficients. By applying the two-stage stochastic homogenization method, the original elliptic equation with random and fast oscillatory coefficients is approximated as an equivalent elliptic equation, where the equivalent coefficients are obtained by solving a set of cell problems. Without the stationary assumption, the number of cell problems is large and the corresponding computational cost is high. To improve the efficiency, we apply the normalizing field flow model to learn a reference Gaussian field for the random equivalent coefficients based on a small amount of data, which is obtained by solving the cell problems with the finite element method. Numerical results demonstrate that the newly proposed method is efficient and accurate in tackling high dimensional partial differential equations in composite materials with complex random microstructures.

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1   Introduction

A wide range of composite materials, such as concrete materials, polymer composites, short-fiber composites, and carbon black reinforced rubber composites, used in science and engineering fields or existed in nature, display aperiodic and random arrangement of the compositions. Evaluating the effective physical and mechanical responses of such materials cannot ignore the spatial randomness that markedly characterizes the configuration of inclusions. Mathematically, the physical and mechanical responses of the random composite materials can be modeled by partial differential equations (PDEs) with random and fast oscillatory coefficients. As a typical example, the random elliptic equations have been widely used in mechanics, heat transfer and electro-magnetics of composite materials (see [1–6]). The physical and mechanical responses of the random composite materials can be obtained by solving those equations. However, due to the microstructure and randomness of the composite materials, it is difficult to solve the random elliptic equations by using the finite element method, because it requires a very fine mesh to resolve the microstructure and large-scale sampling to capture the randomness.

In order to overcome above difficulties, based on the stationary hypothesis, a stochastic homogenization method and corresponding convergence theory were developed in the recent years (see [7–10]). The stochastic homogenization method produces a homogenized solution to the random elliptic equation by solving the homogenized (deterministic) elliptic equation, whose (homogenized) coefficient matrix is computed from the solution to a certain cell problem posed in the entire d-dimensional real vector space. Numerically solving such an infinite domain cell problem is very expensive. To reduce the computational cost, some localized approximations, such as "periodization" or "cut-off" procedures, were introduced in [11, 12]. In the localized approximations, the resulting cell problem is now posed on a bounded domain and it is proved that the approximated coefficients converge to the homogenized coefficients as the size of the bounded domain goes to infinity. Due to the accurate requirement, the bounded domain should be large, which makes the localized approximation inefficiency.

Another popular approach used in scientific and engineering computing, namely the representative volume element (RVE) method, was used to compute the effective parameters of highly heterogeneous materials or random composite materials, in which the cell problem was posed on a reasonably large cell [5, 6, 13–15]. In the RVE method, the size of the cell is in the order $\mathcal{O}(1)$ and usually does not tend to infinity. For the random composite materials, the coefficients in the resulting homogenized problem are still random functions, fluctuating in probability space. After that, under the stationary hypothesis, it was proved that the equivalent matrix can be rewritten as a small random