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Sixth-Order Compact Finite Difference Method for 2D Helmholtz Equations with Singular Sources and Reduced Pollution Effect

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Abstract. Due to its highly oscillating solution, the Helmholtz equation is numerically challenging to solve. To obtain a reasonable solution, a mesh size that is much smaller than the reciprocal of the wavenumber is typically required (known as the pollution effect). High-order schemes are desirable, because they are better in mitigating the pollution effect. In this paper, we present a high-order compact finite difference method for 2D Helmholtz equations with singular sources, which can also handle any possible combinations of boundary conditions (Dirichlet, Neumann, and impedance) on a rectangular domain. Our method is sixth-order consistent for a constant wavenumber, and fifth-order consistent for a piecewise constant wavenumber. To reduce the pollution effect, we propose a new pollution minimization strategy that is based on the average truncation error of plane waves. Our numerical experiments demonstrate the superiority of our proposed finite difference scheme with reduced pollution effect to several state-of-the-art finite difference schemes, particularly in the critical pre-asymptotic region where $\mathsf{k}h$ is near 1 with k being the wavenumber and h the mesh size.

AMS subject classifications: 65N06, 35J05

Key words: Helmholtz equation, finite difference, pollution effect, interface, pollution minimization, mixed boundary conditions, corner treatment.

1 Introduction and motivations

In this paper, we study the 2D Helmholtz equation, which is a time-harmonic wave propagation model, with a singular source term along a smooth interface curve and mixed boundary conditions. The Helmholtz equation appears in many applications such as

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electromagnetism [3,35], geophysics [7,11,15,20], ocean acoustics [31], and photonic crystals [21]. Let $\Omega := (l_1,l_2)\times(l_3,l_4)$ and ψ be a smooth two-dimensional function. Consider a smooth curve $\Gamma := \{(x,y)\in\Omega : \psi(x,y)=0\}$, which partitions Ω into two subregions: $\Omega_+ := \{(x,y)\in\Omega : \psi(x,y)>0\}$ and $\Omega_- := \{(x,y)\in\Omega : \psi(x,y)<0\}$. The model problem (see Fig. 1 for an illustration) is defined as follows:

$$\begin{cases} \Delta u + \mathsf{k}^2 u = f & \text{in } \Omega\setminus\Gamma, \\ [u]=g, \quad [\nabla u\cdot\vec{n}]=g_\Gamma & \text{on } \Gamma, \\ \mathcal{B}_1 u = g_1 \text{ on } \Gamma_1 := \{l_1\}\times(l_3,l_4), & \mathcal{B}_2 u = g_2 \text{ on } \Gamma_2 := \{l_2\}\times(l_3,l_4), \\ \mathcal{B}_3 u = g_3 \text{ on } \Gamma_3 := (l_1,l_2)\times\{l_3\}, & \mathcal{B}_4 u = g_4 \text{ on } \Gamma_4 := (l_1,l_2)\times\{l_4\}, \end{cases} \tag{1.1}$$

where $\partial\Omega = \cup_{i=1}^4 \overline{\Gamma_i}$, k is the wavenumber, f is the source term, and for any point $(x_0,y_0)\in\Gamma$,

$$[u](x_0,y_0) := \lim_{(x,y)\in\Omega_+,(x,y)\to(x_0,y_0)} u(x,y) - \lim_{(x,y)\in\Omega_-,(x,y)\to(x_0,y_0)} u(x,y), \tag{1.2}$$

$$[\nabla u\cdot\vec{n}](x_0,y_0) := \lim_{(x,y)\in\Omega_+,(x,y)\to(x_0,y_0)} \nabla u(x,y)\cdot\vec{n} - \lim_{(x,y)\in\Omega_-,(x,y)\to(x_0,y_0)} \nabla u(x,y)\cdot\vec{n}, \tag{1.3}$$

where $\vec{n}$ is the unit normal vector of Γ pointing towards Ω_+. In (1.1), the boundary operators $\mathcal{B}_1,\cdots,\mathcal{B}_4 \in \{\mathbf{I}_d, \frac{\partial}{\partial\vec{n}}, \frac{\partial}{\partial\vec{n}} - \mathsf{i}\mathsf{k}\mathbf{I}_d\}$, where $\mathbf{I}_d$ corresponds to the Dirichlet boundary condition (sound soft boundary condition for the identical zero boundary datum), $\frac{\partial}{\partial\vec{n}}$ corresponds to the Neumann boundary condition (sound hard boundary condition for the identical zero boundary datum), and $\frac{\partial}{\partial\vec{n}} - \mathsf{i}\mathsf{k}\mathbf{I}_d$ (with i being the imaginary unit) corresponds to the impedance boundary condition. Moreover, the Helmholtz equation of (1.1) with $g=0$ is equivalent to finding the weak solution $u\in H^1(\Omega)$ of $\Delta u + \mathsf{k}^2 u = f + g_\Gamma\delta_\Gamma$ in Ω, where δ_Γ is the Dirac distribution along the interface curve Γ.

The Helmholtz equation is challenging to solve numerically due to several reasons. The first is due to its highly oscillatory solution, which necessitates the use of a very small mesh size h in many discretization methods. Taking a mesh size h proportional to the reciprocal of the wavenumber k is not enough to guarantee that a reasonable solution is obtained or a convergent behavior is observed. The mesh size h employed in a standard discretization method often has to be much smaller than the reciprocal of the wavenumber k. In the literature, this phenomenon is referred to as the pollution effect, which has close ties to the numerical dispersion (or a phase lag). The situation is further exacerbated by the fact that the discretization of the Helmholtz equation typically yields an ill-conditioned coefficient matrix. Taken together, one typically faces an enormous ill-conditioned linear system when dealing with the Helmholtz equation, where standard iterative schemes fail to work [16].

To gain a better insight on how the mesh size requirement is related to the wavenumber, we recall some relevant findings on the finite element method (FEM) and finite difference method (FDM). Two common ways to quantify the pollution effect are through the error analysis and the dispersion analysis. In the FEM literature, the former route