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Invariant Subspace and Exact Solutions to the Generalized Kudryashov-Sinelshchikov Equation

LI Jina[1,*], QU Gaizhu[2], ZHANG Jianlin[1] and JI Xuehui[1]

[1] *College of Science, Zhongyuan University of Technology, Zhengzhou 450007, China.*
[2] *School of Mathematics and Statistics, Weinan Normal University, Weinan 714099, China.*

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Abstract. In this research, invariant subspaces and exact solutions for the governing equation are obtained through the invariant subspace method, and the generalized second-order Kudryashov-Sinelshchikov equation is used to describe pressure waves in a liquid with bubbles. The governing equations are classified and transformed into a system of ordinary differential equations, and the exact solutions of the classified equation are obtained by solving the system of ordinary differential equations. The method gives logarithmic, polynomial, exponential, and trigonometric solutions for equations. The primary accomplishments of this work are displaying how to obtain the exact solutions of the classified equations and giving the stability analysis of reduced ordinary differential equations.

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1 Introduction

Nonlinear science is an interdisciplinary subject that studies the universality characteristics of many nonlinear phenomena in various systems. Its main contents include solitons, chaos, and fractals. Today, scientists have successfully applied soliton theory to many scientific and engineering fields. Partial differential equations (PDEs) are very important because they can describe various reality systems and explain the features of different

*Corresponding author. *Email addresses:* `lijina@zut.edu.cn` (J. N. Li)

scientific fields, such as biology, gas dynamics, nonlinear optics, fluid mechanics, solid-state physics, population modeling, and optics. Studying analytical or exact solutions is important because they show the physical structure of these problems and may provide additional physical information [1-6]. Many physical problems are explained as mathematical models by equations, so analytical or exact solutions play a variety of roles in nonlinear science. The past several decades have witnessed many methods, such as the symmetry group method [7-13], B*ä*cklund transformation [14-15], Darboux transformation [16], inverse scattering method [17], Hirota bilinear method [18], and invariant subspace method, which provide an effective tool for deriving the exact solutions of differential equations [19-34]. The authors studied nonlinear waves in liquid-gas bubble mixtures [35]. Under different assumptions, nonlinear differential equations are obtained [36-38], one of which gives a second-order equation describing pressure waves in a liquid with bubbles. This model is shown by

$$v_t=(1+v)v_{xx}+v_x^2-vv_x, \tag{1.1}$$

the generalization of Eq. (1.1) is also studied [39-42].

The main task of this article is to study the invariant subspace classification and exact solutions of the generalized second-order Kudryashov-Sinelshchikov equation

$$u_t=\tilde{F}(u)=G(u)u_{xx}+F(u)u_x+H(u)u_x^2, \tag{1.2}$$

where u depends on t and x, and $G(u)\neq 0$, $F(u)\neq 0$ and $H(u)\neq 0$ need to be determined as the smooth functions of each of their variables.

The framework of the paper is as follows. In the second section, we show the calculation process of invariant subspaces and all the invariant subspaces of the second-order generalized Kudryashov-Sinelshchikov equation. In the third section, the exact solutions of the equations are constructed by using the invariant subspace method, and stability analysis of the reduced ODEs is shown. Finally, the paper summarizes the research results and provides some opinions.

2 Classification of Eq. (1.2)

The main research work of this part is to classify Eq. (1.2) and to show all the invariant subspace results.

The subspace W_n of Eq. (1.2) is defined as

$$L[y]=y^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_1y'+a_0y=0,$$

where $y^{(n)}=\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ and a_0, $a_1,\cdots$, a_{n-1} are arbitrary constants. Therefore, the differential operator $\tilde{F}(u)$ admits the invariant subspace W_n

$$\left(D^n\tilde{F}+a_{n-1}D^{n-1}\tilde{F}+\cdots+a_0F\right)\Big|_{\tilde{H}}=0.$$