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An Elliptic Nonlinear System of Two Functions with Application

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Abstract. The purpose of this paper is to give a sufficient conditions for the existence and uniqueness of positive solutions to a rather general type of elliptic system of the Dirichlet problem on a bounded domain Ω in R^n. Also considered are the effects of perturbations on the coexistence state and uniqueness. The techniques used in this paper are super-sub solutions method, eigenvalues of operators, maximum principles, spectrum estimates, inverse function theory, and general elliptic theory. The arguments also rely on some detailed properties for the solution of logistic equations. These results yield an algebraically computable criterion for the positive coexistence of competing species of animals in many biological models.

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1 Introduction

One of the prominent subjects of study and analysis in mathematical biology concerns the competition of two or more species of animals in the same environment. Especially pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others. In this paper, we focus on the general competition

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model to better understand the competitive interactions between two species. Specifically, we investigate the conditions needed for the coexistence of two species when the factors affecting them are fixed or perturbed. In earlier literature, the models were concerned with studying those with homogeneous Neumann boundary conditions. Later on, the more important Dirichlet problems, which allow flux across the boundary, became the subject of study.

2 Literature review

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple competition model, commonly known as the Lotka-Volterra competition model. This system describes the competitive interaction of two species residing in the same environment in the following manner:

Suppose two species of animals, rabbits and squirrels for instance, are competing in a bounded domain Ω. Let $u(x,t)$ and $v(x,t)$ be densities of the two habitats in the place x of Ω at time t. Then we have the dynamic competition model

$$\begin{cases} u_t(x,t)=\Delta u(x,t)+au(x,t)-bu^2(x,t)-cu(x,t)v(x,t), \\ v_t(x,t)=\Delta v(x,t)+dv(x,t)-fv^2(x,t)-eu(x,t)v(x,t), & \text{in } \Omega\times[0,\infty), \\ u(x,t)=v(x,t)=0, & \text{for } x\in\partial\Omega, \end{cases}$$

where $a,d>0$ are growth rates, $b,f>0$ are self-limitation rates, and $c,e>0$ are competition rates. Here we are interested in the time independent, positive solutions, i.e. the positive solutions $u(x),v(x)$ of

$$\begin{cases} \Delta u(x)+u(x)(a-bu(x)-cv(x))=0, \\ \Delta v(x)+v(x)(d-fv(x)-eu(x))=0, & \text{in } \Omega, \\ u|_{\partial\Omega}=v|_{\partial\Omega}=0, \end{cases} \tag{2.1}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so its existence means that the two species of animals can live peacefully and forever.

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (2.1) (see [1–7]).

One of the initial important results for the time-independent Lotka-Volterra model was obtained by Cosner and Lazer. In 1984, they published the following sufficient conditions for the existence and uniqueness of a positive steady state solution to (2.1):

Theorem 2.1. (in [2])
(A) *If* $a>\lambda_1+\frac{cd}{f}$, $d>\lambda_1+\frac{ae}{b}$, *where* λ_1 *is the smallest eigenvalue of* $-\Delta$ *with homogeneous boundary conditions as in the Lemma* 3.2, *then there exist positive smooth functions u and v in* Ω *satisfying* (2.1).