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Quasi Contraction of Stochastic Functional Differential Equations

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Abstract. Using a novel approach, we present explicit criteria for the quasi contraction of stochastic functional differential equations. As an application, some sufficient conditions ensuring the contraction property of the solution to the considered equations are obtained. Finally, some examples are investigated to illustrate the theory.

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1 Introduction

Recently, the contraction problem of stochastic differential equations has attracted lots of attention and become one of the most active areas in biology [1], control theory [2], observer design [3], synchronization of coupled oscillators [4], traffic networks [5], and so on. For example, Dahlquist [6] employed logarithmic norms to demonstrate the contractivity of differential equations. Aminzare et al. [7] investigated nonlinear system contraction methods. The contraction analysis for hybrid systems was studied by Burden et al. [8]. Margaliot et al. [9] proposed three generalizations of contraction based on a norm that allows contraction to take place after small transients in time or amplitude.

The Lyapunov function method is a well-known method for determining contractibility in stochastic differential equations. Contractibility of stochastic differential equations has been achieved using Lyapunov functions and functionals (see [10–12]). For stochastic differential equations, finding a Lyapunov function is difficult, and the contractibility criteria produced by the Lyapunov function approach are frequently expressed in

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terms of differential inequalities, matrix inequalities, and so on. The Lyapunov function's stated requirements are not only a little bit strong, but also broadly implicit and difficult to investigate. Furthermore, when studying the contraction of stochastic differential equations by integral inequalities, we find that there are two flaws: the coefficients must typically satisfy the Lipschitz condition, and the Lipschitz constants must usually be sufficiently small.

In this paper, we will investigate quasi contraction of stochastic differential equations using a novel approach that does not require any integral inequality. Furthermore, We establish the explicit exponential contraction condition for stochastic differential equations. Using Itô's formulae, we will get some new sufficient conditions ensuring the contraction of stochastic differential equations based on a comparison principle and proof by reductio ad absurdum.

The following is how the rest of the paper is structured: We introduce some necessary notations and preliminaries in Section 2. The quasi contraction and exponential contraction of stochastic differential equations are discussed in Section 3. In Section 4, we give some instances to show how our findings are beneficial.

2 Preliminary

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let H, K be two real separable Hilbert spaces and we denote by $\langle\cdot,\cdot\rangle_H$, $\langle\cdot,\cdot\rangle_K$ their inner products and by $\|\cdot\|_H$, $\|\cdot\|_K$ their vector norms, respectively. We denote by $\mathcal{L}(K,H)$ the set of all linear bounded operators from K into H, equipped with the usual operator norm $\|\cdot\|$. Let $\tau>0$ and $C:=C([-\tau,0];H)$ denote the family of all continuous functions from $[-\tau,0]$ to H. The space $C([-\tau,0];H)$ is assumed to be equipped with the norm $\|\varphi\|_C=\sup_{-\tau\leq\theta\leq 0}\|\varphi(\theta)\|_H$. We also denote $C^b_{\mathcal{F}_0}([-\tau,0];H)$ be the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $C([-\tau,0];H)$-valued random variables.

Let $\{W(t),t\geq 0\}$ denote a K-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Wiener process defined on $(\Omega,\mathcal{F},\mathbb{P})$ with covariance operator Q, i.e.,

$$\mathbb{E}\langle W(t),x\rangle_K\langle W(s),y\rangle_K=(t\wedge s)\langle Qx,y\rangle_K \qquad \text{for all } x,y\in K,$$

where Q is a positive, self-adjoint, trace class operator on K. In particular, we shall call such $W(t),t\geq 0$, a K-valued Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

In order to define stochastic integrals with respect to the Q-Wiener process $W(t)$, we introduce the subspace $K_0=Q^{1/2}(K)$ of K which, endowed with the inner product

$$\langle u,v\rangle_{K_0}=\langle Q^{-\frac{1}{2}}u,Q^{-\frac{1}{2}}v\rangle_K$$

is a Hilbert space. Let $\mathcal{L}_2^0=\mathcal{L}_2(K_0,H)$ denote the space of all Hilbert-Schmidt operators