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Properties of Solutions to Fractional Laplace Equation with Singular Term

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Abstract. The aim of the paper is to study the properties of positive classical solutions to the fractional Laplace equation with the singular term. Using the extension method, we prove the nonexistence and symmetric of solutions to the singular fractional equation.

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1 Introduction

In this paper, we consider the fractional problem

$$(-\Delta)^{\frac{\alpha}{2}}u=-u^p, \quad \text{in } \mathbb{R}^n \ (n\geq 3), \tag{1.1}$$

where $1\leq\alpha<2$, $-\infty<p\leq\frac{n+2-\alpha}{n-\alpha}$. The fractional Laplacian $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^n$ is nonlocal pseudo-differential operator defined by

$$(-\Delta)^{\frac{\alpha}{2}}u(x)=C_{n,\alpha}PV\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}\mathrm{d}y,$$

where PV is the Cauchy principal value, $C_{n,\alpha}$ is a positive constant. For the information about the fractional Laplacian can refer to [1,2].

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Caffarelli and Silvestre in [3] put forward the extension method which can reduce the nonlocal problem relating to $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^n$ to a local one in higher dimensions in $\mathbb{R}^{n+1}_+$. For a function $u:\mathbb{R}^n\to\mathbb{R}$, define the extension $U:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ that satisfies

$$\begin{cases}\operatorname{div}(y^{1-\alpha}\nabla U)=0, & (x,y)\in\mathbb{R}^n\times[0,\infty),\\ U(x,y)=u(x), & x\in\mathbb{R}^n,\ y=0,\end{cases}$$

they showed that

$$(-\Delta)^{\frac{\alpha}{2}}u(x)=-C_{n,\alpha}\lim_{y\to 0^+}y^{1-\alpha}\frac{\partial U}{\partial y}.$$

This method has been applied to deal with equations involving the fractional Laplacian and fruitful results have been obtained, see [4–7] and the references therein.

We use the above extension method to reduce the nonlocal problem (1.1) into a local one and apply the method of moving planes to the local problem

$$\begin{cases}\operatorname{div}(y^{1-\alpha}\nabla U)=0, & (x,y)\in\mathbb{R}^n\times[0,\infty),\\ \dfrac{\partial U}{\partial \nu^\alpha}=-u^p(x,0), & x\in\mathbb{R}^n,\end{cases}\tag{1.2}$$

where

$$\frac{\partial U}{\partial \nu^\alpha}:=-\lim_{y\to 0^+}y^{1-\alpha}\frac{\partial U}{\partial y}.$$

There are many results about the Laplace equation with nonlinear boundary conditions problem

$$\begin{cases}\Delta u=0, & \text{in } \mathbb{R}^n_+,\\ D_{x_n}u=u^p, & \text{on } \partial\mathbb{R}^n_+.\end{cases}\tag{1.3}$$

In [8], Hu proved that (1.3) has no positive solution for $1\le p<\frac{n}{n-2}$. This nonexistence result have been extended to $p<\frac{n}{n-2}$ by Ou in [9], and the form of solution for $p=\frac{n}{n-2}$ also has been obtained. Zhu considered the relative problem of (1.3) in [10], given the Liouville type theorems about the nonnegative solution to some indefinite elliptic equations. In [11], Lou and Zhu shown the classification of the solutions of problem (1.3). Their ideas lead us to study the problem (1.1) by its extension problem (1.2). The main method in those above papers is the method of moving planes which comes from Gidas, Ni and Nirenberg [12], and then this method have been to handle to many other types Laplace equations to get their properties of the solutions, can refer to [13–16] and their references therein. Cai and Ma [17] studied the radially symmetry properties of positive solutions to the fractional Laplace equation with negative powers, which add the growth/decay property of the solution. In this paper, by the extension method the condition can be delete.

Our main results are the following: