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New Sixth-Order Compact Schemes for Poisson/Helmholtz Equations

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Abstract. Some new sixth-order compact finite difference schemes for Poisson/ Helmholtz equations on rectangular domains in both two- and three-dimensions are developed and analyzed. Different from a few sixth-order compact finite difference schemes in the literature, the finite difference and weight coefficients of the new methods have analytic simple expressions. One of the new ideas is to use a weighted combination of the source term at staggered grid points which is important for grid points near the boundary and avoids partial derivatives of the source term. Furthermore, the new compact schemes are exact for 2D and 3D Poisson equations if the solution is a polynomial less than or equal to 6. The coefficient matrices of the new schemes are M-matrices for Helmholtz equations with wave number $K \leq 0$, which guarantee the discrete maximum principle and lead to the convergence of the new sixth-order compact schemes. Numerical examples in both 2D and 3D are presented to verify the effectiveness of the proposed schemes.

AMS subject classifications: 65N06, 65N12

Key words: Poisson equation, Helmholtz equation, sixth-order compact scheme, maximum principle, staggered grid.

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1. Introduction

In [5], a new strategy for constructing high-order compact finite difference schemes for second-order elliptic partial differential equations (PDEs) has been studied. Several fourth- and third-order compact schemes are developed for different PDEs with flux-type boundary conditions. In this study, we extend the strategy from [5] with some

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new ideas to develop sixth-order compact (SOC) finite difference schemes for Poisson/Helmholtz equations,

$$\Delta u + Ku = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.1}$$

$$u = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{1.2}$$

where Ω is a rectangular domain in 2D, and a cubic domain in 3D, and K is a constant. How to solve Poisson/Helmholtz equations accurately and efficiently is a kernel part of many applications. High order compact schemes (HOC) are especially useful for high wave number computations, oscillatory solutions, and problems on infinite domains. There are many types of research on fourth-order compact schemes for Poisson/Helmholtz equations, please see [6, 10, 12] and references therein.

However, there are a few sixth-order ones in the literature for 2D problems. Based on the Padé approximation, a class of sixth-order schemes is discussed in [8, 9, 11]. By combining Simpson integral formula and parabolic interpolation, authors in [18] studied fourth-order and sixth-order compact schemes of Poisson equations. An optimal compact finite difference scheme for solving Helmholtz equations with a refined optimization rule for choosing the weight parameters is presented in [16]. A strategy to make some sixth-order schemes less sensitive to wave numbers for Helmholtz equations was proposed in [4].

For 3D problems, a couple of sixth-order compact schemes are developed in [13, 14]. Some existing SOC schemes were restricted to constant K, the authors extend these compact sixth-order schemes to variable $K(\mathbf{x})$ in both two- and three-dimensions in [15]. In [17], using a finite volume method, the authors derived a family of SOC schemes for Poisson equations based on dual partitions. In [1], the authors have presented a method using symmetric polynomial algebra to compute the coefficients of some SOC schemes for Poisson equations. Recently, a SOC scheme for Poisson equations with discontinuous or singular source terms has been developed in [2].

As we can see from the literature, most SOC schemes need to compute finite difference coefficients and weights for source terms as part of algorithms and require some partial derivatives of the source term f or their fourth-order approximations. This study extends the work in [5] to obtain SOC finite difference schemes for Poisson/Helmholtz equations both in 2D and 3D with Dirichlet boundary conditions. The extension is nontrivial, and the new idea is to use a staggered grid for the source terms. The distinguishing feature of the new SOC schemes includes: there are analytic expressions of the finite difference coefficients and the weights for source terms; no derivatives of source terms are needed; the coefficient matrices are M-matrix which guarantee the discrete maximum principle and the convergence of the algorithms; the new schemes are exact if the solution is any sixth-order polynomial.

The rest of the paper is organized as follows. In Section 2, we derive the SOC schemes for 2D Poisson/Helmholtz equations with Dirichlet boundary conditions. In Section 3, we give the SOC schemes for 3D Poisson/Helmholtz equations. In Section 4, we present the numerical results of our SOC schemes. We conclude in the last section.