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On a Class of Quasilinear Elliptic Equations

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Abstract. We consider a class of quasilinear elliptic boundary problems, including the following Modified Nonlinear Schrödinger Equation as a special case:

$$\begin{cases} \Delta u + \dfrac{1}{2} u\Delta(u^2) - V(x)u + |u|^{q-2}u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is the entire space $\mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $q \in (2, 22^*]$ with $2^* = 2N/(N-2)$ being the critical Sobolev exponent and $22^* = 4N/(N-2)$. We review the general methods developed in the last twenty years or so for the studies of existence, multiplicity, nodal property of the solutions within this range of nonlinearity up to the new critical exponent $4N/(N-2)$, which is a unique feature for this class of problems. We also discuss some related and more general problems.

AMS subject classifications: 35B08, 35B20, 35B33, 35J20, 35J62, 35J92, 35Q55, 35P30

Key words: Variational perturbations, p-Laplacian regularization, quasilinear elliptic equations, modified nonlinear Schrödinger equations, sign-changing solutions, critical exponents.

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1 Introduction

We consider quasilinear equations of the following form which have been studied extensively for various aspects of the problems during the last two decades

$$\begin{cases} \sum_{i,j=1}^{N} D_j\big(a_{ij}(x,u)D_iu\big) - \frac{1}{2}\sum_{i,j=1}^{N} D_s a_{ij}(x,u)D_iuD_ju \\ \qquad -V(x)u+f(x,u)=0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded or unbounded domain with smooth boundary, or $\Omega=\mathbb{R}^N$ the entire space. We use the notations

$$D_iu=\frac{\partial u}{\partial x_i},\quad D_s a_{ij}(x,s)=\frac{\partial}{\partial s}a_{ij}(x,s).$$

When we work in bounded domains we often ignore the linear potential term as the results are the same with or without V when the potential V has a positive lower bound.

A special and important class of equations from (1.1) is the following so-called modified nonlinear Schrödinger equations (MNLS):

$$\begin{cases} \Delta u - V(x)u+\frac{1}{2}u\Delta u^2+|u|^{q-2}u=0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which correspond to the case of $a_{ij}(x,s)=(1+s^2)\delta_{ij}, f(x,s)=|s|^{q-2}s$. This equation and its extensions have been involved in many models of mathematical physics and have received considerable attention during the last years, we refer to [12, 43, 44, 49, 50, 53, 64] and the references therein for some physics related literature. Mathematically the problems have been studied in recent years both for the stationary and evolutionary problems. For evolutionary problems we refer e.g., [28,30,48,52] and references therein. We will review mainly on stationary problems in this paper, in particular on the existence and multiple existence results.

For the special case of interest (MNLS)

$$a_{ij}(x,u)=(1+u^2)\delta_{ij}$$

the weak formulation is to find $u\in H_0^1\cap L^\infty$ such that

$$\int_\Omega \Big[(1+u^2)\nabla u\nabla\phi+u|\nabla u|^2\phi+Vu\phi-f(x,u)\phi\Big]dx=0,\quad \forall\phi\in C_0^\infty(\Omega). \tag{1.3}$$