Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2021-0184

Vol. **16**, No. 1, pp. 242-276
February 2023

The Direct Method of Lines for Forward and Inverse Linear Elasticity Problems of Composite Materials in Star-shaped Domains

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Received 3 December 2021; Accepted (in revised version) 10 July 2022

Abstract. In this paper, we generalize the direct method of lines for linear elasticity problems of composite materials in star-shaped domains and consider its application to inverse elasticity problems. We assume that the boundary of the star-shaped domain can be described by an explicit C^1 parametric curve in the polar coordinate. We introduce the curvilinear coordinate, in which the irregular star-shaped domain is converted to a regular semi-infinite strip. The equations of linear elasticity are discretized with respect to the angular variable and we solve the resulting semi-discrete approximation analytically using a direct method. The eigenvalues of the semi-discrete approximation converge quickly to the true eigenvalues of the elliptic operator, which helps capture the singularities naturally. Moreover, an optimal error estimate of our method is given. For the inverse elasticity problems, we determine the Lamé coefficients from measurement data by minimizing a regularized energy functional. We apply the direct method of lines as the forward solver in order to cope with the irregularity of the domain and possible singularities in the forward solutions. Several numerical examples are presented to show the effectiveness and accuracy of our method for both forward and inverse elasticity problems of composite materials.

AMS subject classifications: 65N21, 65N40, 74A40, 74B05

Key words: Composite materials, linear elasticity problems, inverse elasticity problems, star-shaped domains, method of lines.

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1. Introduction

With the growing application of composite materials, the linear elasticity problem

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of composite materials has drawn a great deal of attention from mathematicians and engineers. In this paper, we focus on both forward and inverse linear elasticity problems of composite materials in star-shaped domains. We first describe and make some assumptions on the geometry of the star-shaped domain $\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k \subset \mathbb{R}^2$, where Ω stands for the whole material composed of K kinds of different materials and Ω_k stands for the k-th kind of material. We assume that the boundary $\Gamma = \partial\Omega$ is star-shaped with respect to the origin O and can be described by an explicit C^1 parametric curve in the polar coordinates. To be more precise, we assume that Γ can be parameterized as a (piecewise) C^1 function of the angular variable ϕ, denoted by $\tilde{r}(\phi)$, such that $\tilde{r}(0) = \tilde{r}(2\pi)$ and $\tilde{r}(\phi) \geq r_0 > 0$ for any $0 \leq \phi \leq 2\pi$. This assumption on the geometry of Γ is made so that the curvilinear coordinate which will be introduced in (2.1) is well defined; see also (2.3)-(2.11). Without loss of generality, we also assume that all the interfaces between different materials meet at the origin and each interface $\overline{\Omega}_{k-1} \cap \overline{\Omega}_k$ is a line segment

$$L_k = \big\{(r,\theta)\,|\,\theta = \theta_k,\, 0 \leq r \leq \tilde{r}(\theta_k)\big\} \quad \text{for} \quad k = 1,\dots,K,$$

where $\Omega_0 = \Omega_K, \theta_1 = 0$ (See Fig. 1).

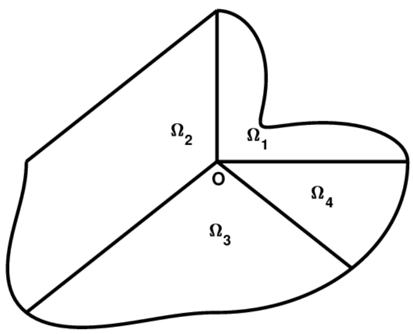

Figure 1: Composite materials in a star-shaped domain.

We also let $\theta_{K+1} = 2\pi$. We consider the following Navier's equations in Ω with Dirichlet boundary conditions, which describe the equilibrium state in linear elasticity:

$$\begin{aligned} & -\nabla \cdot \sigma^k = 0 && \text{in } \Omega_k, \\ & u^k = f^k && \text{on } \Gamma_k, \\ & u^{k-1}|_{\theta=\theta_k^-} = u^k|_{\theta=\theta_k^+}, && \\ & \sigma^{k-1} \cdot n_k|_{\theta=\theta_k^-} = \sigma^k \cdot n_k|_{\theta=\theta_k^+} && \end{aligned} \tag{1.1}$$

with

$$\sigma^k = 2\mu_k \varepsilon(u^k) + \lambda_k \nabla \cdot u^k I, \quad \varepsilon(u^k) = \frac{1}{2}\big(\nabla u^k + (\nabla u^k)^T\big), \tag{1.2}$$

where $k = 1,\dots,K$, $\Gamma_k = \partial\Omega_k \bigcap \partial\Omega$, f is a given vector-valued function on Γ

$$f|_{\Gamma_k} = f^k = \big(f_1^k, f_2^k\big)^T,$$