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Error Estimate of a New Conservative Finite Difference Scheme for the Klein-Gordon-Dirac System

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Abstract. In this paper, we derive and analyze a conservative Crank-Nicolson-type finite difference scheme for the Klein-Gordon-Dirac (KGD) system. Differing from the derivation of the existing numerical methods given in literature where the numerical schemes are proposed by directly discretizing the KGD system, we translate the KGD equations into an equivalent system by introducing an auxiliary function, then derive a nonlinear Crank-Nicolson-type finite difference scheme for solving the equivalent system. The scheme perfectly inherits the mass and energy conservative properties possessed by the KGD, while the energy preserved by the existing conservative numerical schemes expressed by two-level's solution at each time step. By using energy method together with the 'cut-off' function technique, we establish the optimal error estimate of the numerical solution, and the convergence rate is $\mathcal{O}(\tau^2 + h^2)$ in l^∞-norm with time step τ and mesh size h. Numerical experiments are carried out to support our theoretical conclusions.

AMS subject classifications: 65M06, 65M12

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1. Introduction

In this paper, we consider the Klein-Gordon-Dirac (KGD) equation in d ($d = 1, 2$) dimensions [17]

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$$\begin{aligned} &\partial_{tt}\phi(t,\mathbf{x}) - \Delta\phi(t,\mathbf{x}) + \phi(t,\mathbf{x}) \\ &= g\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ &i\partial_t\Psi(t,\mathbf{x}) + i\sum_{j=1}^{d}\sigma_j\partial_j\Psi(t,\mathbf{x}) - \omega\sigma_3\Psi(t,\mathbf{x}) \\ &= g\phi(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^d, \quad t>0. \end{aligned} \tag{1.1}$$

The Klein-Gordon-Dirac equation is a fundamental model in quantum electrodynamics and describes the dynamics of a complex-value Dirac vector field $\Psi(t,\mathbf{x}) := (\psi_1(t,\mathbf{x}),\psi_2(t,\mathbf{x})) \in \mathbb{C}^2$ interacting with a neutral real-valued meson field $\phi := \phi(t,\mathbf{x}) \in \mathbb{R}$ through the Yukawa potential [13,25,27,33,36] with a coupling constant $0 < g \in \mathbb{R}$. Note that $\omega > 0$ is the radio between the mass of the electron and the mass of the meson, $i = \sqrt{-1}$ is the imaginary unit, t is time, $\mathbf{x}\in\mathbb{R}^d$ is the spatial coordinate vector. In 2D, $\mathbf{x} = (x_1,x_2)^T$, $\partial_j = \partial/\partial x_j$, $j = 1,2$ and $\Delta = \sum_{j=1}^2 \partial_j^2$. In addition, Ψ^* is the conjugate transpose of Ψ, σ_1, σ_2 and σ_3 are the Pauli matrices given as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

The KGD system (1.1) is dispersive and conserves the total mass

$$M(t) := \|\Psi(t,\cdot)\|_{L^2}^2 = \int_{\mathbb{R}^d} |\Psi(t,\mathbf{x})|^2 d\mathbf{x} \equiv \|\Psi(0,\cdot)\|_{L^2}^2, \tag{1.3}$$

and energy

$$\begin{aligned} E(t) :=& \frac{1}{2}\int_{\mathbb{R}^d} |\partial_t\phi(t,\mathbf{x})|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} |\partial_x\phi(t,\mathbf{x})|^2 d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^d} |\phi(t,\mathbf{x})|^2 d\mathbf{x} \\ &+ \int_{\mathbb{R}^d} \bigg[i\Psi^*(t,\mathbf{x})\sum_{j=1}^{d}\sigma_j\partial_j\Psi(t,\mathbf{x}) - \omega\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}) \\ &\qquad - g\phi(t,\mathbf{x})\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}) \bigg] d\mathbf{x} \equiv E(0), \quad t\geq 0. \end{aligned} \tag{1.4}$$

For the KGD system (1.1), we introduce an auxiliary function $u := \partial_t\phi$ to rewrite it into the following equivalent system:

$$\begin{aligned} &\partial_t u(t,\mathbf{x}) - \Delta\phi(t,\mathbf{x}) + \phi(t,\mathbf{x}) = g\Psi^*(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}), && \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ &\partial_t\phi(t,\mathbf{x}) = u(t,\mathbf{x}), && \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ &i\partial_t\Psi(t,\mathbf{x}) + i\sum_{j=1}^{d}\sigma_j\partial_j\Psi(t,\mathbf{x}) - \omega\sigma_3\Psi(t,\mathbf{x}) && \\ &= g\phi(t,\mathbf{x})\sigma_3\Psi(t,\mathbf{x}), && \mathbf{x}\in\mathbb{R}^d, \quad t>0. \end{aligned} \tag{1.5}$$