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A Fixed Point Theorem for a Pair of Generalized Nonexpansive Mappings in Uniformly Convex Metric Spaces

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Abstract. In uniformly convex metric spaces, we study the existence and uniqueness of a common fixed point for a pair of generalized nonexpansive mappings with some weak conditions. Meanwhile, we introduce a new Krasnoselskii type iterative algorithm for approximating the common fixed point. A numerical example is also given to demonstrate the main result. Our results generalize and improve some recent corresponding results.

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Key words: Common fixed point, a pair of generalized nonexpansive mappings, Krasnoselskii type iterative algorithm, uniformly convex metric spaces.

1 Introduction and preliminaries

In the past few decades, as generalization of the Banach fixed-point theorem, many authors [1-5,16-23] have defined different nonexpansive and contractive type mappings in the framework of metric spaces and Banach spaces. They all focused on finding suitable conditions for the existence, uniqueness and convergence of fixed points for these mappings. In uniformly convex Banach spaces, Shimi [6] considered a generalized nonexpansive mapping and proved the following result:

Theorem 1.1. *Let X be a uniformly convex Banach space, K be a nonempty, bounded, closed and convex subset of X. Suppose T is a continuous mapping from K into a compact subset of K such that*

$$||Tx-Ty|| \le a||x-y||+b[||x-Tx||+||y-Ty||]+c[||x-Ty||+||y-Tx||] \tag{1.1}$$

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for all $x,y\in K$, *where* $a,b,c\geq 0$ *and* $a+2b+2c\leq 1$. *Let* x_0 *be an arbitrary point of* K. *Then the sequence* $\{x_n\}$ *defined by*

$$x_{n+1}=\frac{1}{2}(x_n+Tx_n),\qquad n=0,1,\cdots,$$

converges to a fixed point of T.

Takahashi [9] first introduced a definition of convex metric space. In fact, every linear normed space and its convex subset are special examples of convex metric spaces. And then, Shimizu and Takahashi [10] gave a notion of uniformly convex structure in a convex metric space, and a convex metric space together with a uniformly convex structure is called a uniformly convex metric space. Very recently, some known fixed point results in uniformly convex Banach spaces have been extended to the case of uniformly convex metric spaces in [11-14].

As a generalization of (1.1), a pair of generalized nonexpansive mappings are defined in a metric space as follows:

Definition 1.1. *Let* K *be a nonempty subset of a metric space* (X,d). *Two mappings* $T,S:K\to K$ *are said to be a pair of generalized nonexpansive mappings if*

$$d(Tx,Sy)\leq ad(x,y)+b[d(x,Tx)+d(y,Sy)]+c[d(x,Sy)+d(y,Tx)] \tag{1.2}$$

for all $x,y\in K$.

In complete metric spaces, common fixed point theorems for (1.2) have been proved by Husain and Sehgal [7] with the condition $a+2b+2c<1$. If K is a closed subset of a uniformly convex Banach space, Bose [8] proved some common fixed point theorems for (1.2) and used the Picard iteraion to approach a common fixed point of (1.2).

Inspired by the above results, this paper aims at giving some sufficient conditions for existence of common fixed points for (1.2) in uniformly convex metric spaces and uniformly convex Banach spaces. And a family of Krasnoselskii type iterations are given to approach a unique common fixed point of (1.2). Our results generalize and improve the corresponding results in [3-8, 11-13].

We need the following definitions and conclusions in our proof.

Definition 1.2 ([9]). *A convex structure in a metric space* (X,d) *is a mapping* $W:X\times X\times[0,1]\to X$ *satisfying, for each* $x,y,u\in X$ *and* $\lambda\in[0,1]$,

$$d(u,W(x,y;\lambda))\leq\lambda d(u,x)+(1-\lambda)d(u,y).$$

A metric space (X,d) *(a Riemannian manifold) together with a convex structure* W *is called a convex metric space* (X,d,W). *Moreover, a nonempty subset* K *of* X *is said to be convex if* $W(x,y;\lambda)\in K$ *for all* $x,y\in K$ *and* $\lambda\in[0,1]$.