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Monotone Iterative Technique for S-Asymptotically Periodic Problem of Evolution Equation with Delay

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Abstract. This paper is concerned with S-asymptotically periodic problem of evolution equation with delay in ordered Banach space. Under some weaker assumptions, we construct monotone iterative method in the presence of the lower and upper solutions to the delayed evolution equation, and obtain the existence of maximal and minimal S-asymptotically periodic mild solutions. Finally, we give an example to exhibit the practicability of our abstract results.

AMS subject classifications: 47J35, 34K30, 34K13, 47D06, 47D60

Key words: Evolution equation, delay, S-asymptotically periodic mild solutions, monotone iterative technique, positive C_0-semigroup.

1 Introduction

Let $(X,\|\cdot\|)$ be the ordered Banach space, whose positive cone $K=\{u\in X|u\geq\theta\}$ is normal with normal constant N, θ is the zero element of X, $r>0$ is a constant. Let $\mathcal{B}:=C([-r,0],X)$ denote the space of continuous functions from $[-r,0]$ into X provided with the uniform norm $\|\phi\|_{\mathcal{B}}=\sup_{s\in[-r,0]}\|\phi(s)\|$. Define a positive cone $K_{\mathcal{B}}$ by $K_{\mathcal{B}}=\{\phi\in\mathcal{B}|\phi(t)\geq\theta,t\in[-r,0]\}$, then $\mathcal{B}$ is an ordered Banach spaces with the partial order relation "$\leq$" induced by the cone $K_{\mathcal{B}}$, and $K_{\mathcal{B}}$ is normal with the normal constant N. If $u:[-r,\infty)\to X$ is a continuous bounded function, then $u_t\in\mathcal{B}$ for each $t\geq 0$, where u_t defined by $u_t(s):=u(t+s)$ for $s\in[-r,0]$. In this article, we discuss the following delayed evolution equation (DEE)

$$\begin{cases} u'(t)+Au(t)=F(t,u(t),u_t), & t\geq 0,\\ u(t)=\varphi(t), & t\in[-r,0], \end{cases} \tag{1.1}$$

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where $A:D(A)\subset X\to X$ is a closed linear operator, and $-A$ generates a C_0-semigroup $T(t)$ $(t\geq 0)$ in X; $F:\mathbb{R}^+\times X\times\mathcal{B}\to X$ is a given function which will be specified later; $\varphi\in\mathcal{B}$.

The periodic problems for evolution equations are widely used in physics, chemistry, information, engineering and other fields. Especially, periodic problems of evolution equations with delay have attracted increasing attention in recent years, see [14, 15, 17, 27, 28, 38] and the references therein. But in the actual development process, due to the influence of external factors, the motion regulations of things often show approximate periodic phenomena in some ways. In order to study these approximate periodic phenomena, the approximate periodic problems for differential equations are often used to construct mathematical models to describe these motion regulations. Therefore, it is very important to study the approximate periodic solutions of differential equations, such as almost periodic solutions, asymptotic periodic solutions, asymptotic almost periodic solutions, pseudo almost periodic solutions and S-asymptotic periodic solutions, see [1,3,12,19,22,37,39].

In particular, in 2008, Henríquez *et al.* [21] first proposed the concept of S-asymptotically ω-periodic function. The relationship between S-asymptotically ω-periodic function and asymptotically ω-periodic function was compared, and the existence of S-asymptotically ω-periodic solutions of abstract Cauchy problems in Banach space was obtained. Later, by using the Leary-Schauder alternative theorem, the existence and uniqueness of the S-asymptotically ω-periodic solutions of the Cauchy problems was obtained, see [2]. There are some papers about S-asymptotically periodic solutions, one can refer to [3,11,12,20,25,26,32–35].

The monotone iterative method based on lower and upper solutions is an effective and flexible mechanism. It yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. Early on, Du and Lakshmikantham [13], Sun and Zhao [36] investigated the existence of extremal solutions to initial value problem of ordinary differential equations by using the method of lower and upper solutions and monotone iterative technique. Later, Li [30] applied lower and upper solutions method to periodic solution problems for semilinear evolution equations without delay in abstract spaces, and obtained the existence of maximal and minimal periodic mild solutions by using the characteristics of positive operators semigroups and the monotone iteration scheme. For the abstract evolution equations, there are more results involving monotone iterative techniques and operator semigroups theory, we can see [4–10,24]. However, as far as we know, there are few results for evolution equations S-asymptotically periodic problems with delay by using the method of the lower and upper solutions coupled with the monotone iterative technique.

Motivated by the papers mentioned above, the purpose of this paper is to construct the general principle for lower and upper solutions coupled with the monotone iterative technique for evolution equations S-asymptotically periodic problems with delay, and obtain the existence of maximal and minimal periodic mild solutions. In the next section, some notions, definitions, and preliminary facts that we need are provided.