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Efficient Spectral-Galerkin Method for Pricing Asian Options

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Abstract. This paper is concerned with a high order numerical method for evaluating the prices of Asian options with fixed strike price. We apply Legendre-Galerkin spectral method for spatial discretization and Crank-Nicholson scheme for temporal discretization. We prove that the scheme is stable for the time discretization, second order accurate in time, and exponential order convergence in space. Numerical experiments are carried out to demonstrate the efficiency and accuracy of the proposed method.

AMS subject classifications: 65M70, 65M12

Key words: Asian option, spectral-Galerkin method, Legendre polynomial, error analysis.

1 Introduction

Asian options, also known as average options, are quite popular in the financial market. These options allow holders to buy the underlying asset at the average price instead of the spot price. Compared to standard options (American and European) where the payoff depends on the value of underlying asset price at a specific point in time, Asian options have the feature that their payoff depends on the average value of underlying asset price over a specified period of time. In general, the Asian option pricing model is a two-dimensional PDE [13]. This PDE has no diffusion in one coordinate direction, and when the volatility σ or spatial variable S is small, the partial differential operator becomes a convection dominated operator. Due to this feature, Asian options are difficult to price compared to standard options.

In some cases, Asian options can be modeled using one-dimensional PDE. Ingersoll [13] showed that two-dimensional floating strike Asian options can be transformed to one-dimensional PDE. Rogers and Shi [17] derived a one-dimensional PDE that can model both floating and fixed strike Asian options. In addition, Večeř [23] showed that Asian

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options are options on a traded account, and characterized Asian options by an alternative one-dimensional PDE. Reduction of spatial dimension for Asian options makes PDE simpler and less computational cost. However, this PDE is difficult to solve numerically since for small value of volatility σ the diffusion term is also small. Now we briefly review some numerical methods for the one-dimensional PDE. Using the change of variables defined in [2], the one-dimensional PDE can be converted from the semi-infinite space domain to a finite space domain. For example, an explicit/implicit finite difference method was applied in [15] to solve the PDE. In [14], a numerical study of the PDE was presented with radial basis functions based on finite differences method. An unconditionally stable compact finite difference scheme was proposed in [16] to solve the one-dimensional PDE. On the other hand, the semi-infinite space domain can be truncated into $(0,L)$ with an appropriate choice of L. For example, a hybrid finite difference method was applied in [6] to obtain numerical solution of the problem. A fourth order numerical method based on B-spline functions was presented in [18]. In [5], a finite difference scheme with a moving mesh was proposed to solve Večeř's PDE.

Most numerical methods of valuing financial options focus on finite difference method and finite element method. Spectral method and spectral element method are limited used in computational finance. To the best of our knowledge, spectral element method was applied in [7,8,25–27] to solve the European option and the American option pricing models. However, there is no literature about spectral method for pricing Asian options. In this paper, we will use efficient Legendre-Galerkin spectral method for numerical approximations of the Asian option pricing model. Compared to finite difference method and finite element method, spectral method is capable of providing superior accuracy with fewer unknowns if the solutions are sufficiently smooth [4,21,22]. In addition, when using spectral method for the option pricing model, the first and second derivatives of the option price can be computed directly and accurately from the spectral representation.

The paper is organized as follows. In Section 2, we introduce the Asian option pricing model. In Section 3, we present time discretization using Crank-Nicholson scheme, and prove the stability and convergence for the time-discrete problem. In Section 4, we present spatial discretization using Spectral-Galerkin method, and carry out a rigorous error estimate for the fully discrete problem. Detailed numerical implementation of the problem is given in Section 5. Numerical results and related discussions are presented in Section 6 to confirm the theoretical analysis. Some concluding remarks are given in the last section.

2 Mathematical model

Let $\Omega_\infty = \{(S,I,t) \in (0,+\infty)\times(0,+\infty)\times[0,T)\}$, where $S=S(t)$ denotes the price of underlying asset, $I=I(t)$ is the underlying asset price running sum defined by

$$I(t)=\int_0^t S(\xi)d\xi,$$