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Offsets of Cassini ovals

Thierry (Noah) Dana-Picard
ndp@jct.ac.il
Jerusalem College of Technology
Havaad Haleumi Street 21
9116011 Jerusalem, Israel

Zoltán Kovács
zoltan@geogebra.org
The Priv. Univ. Coll. of Educ.
of the Diocese of Linz
Salesianumweg 3
4020 Linz, Austria

Abstract

We report on computing and visualizing the offsets of the Cassini ovals. We use the elimination technique to obtain offsets as envelopes of a family of planar curves in a recent experimental version of GeoGebra. Offset curves are obtained in their implicit form as irreducible polynomials of degree 12 and 16.

1 Introduction

The study of plane algebraic curves is a domain which has been explored for centuries. Numerous results on the topic, description of curves form different points of view, are widely available, either in books such as [4] or in websites, for example `mathcurve.com` or `https://mathshistory.st-andrews.ac.uk/Curves/`. Up to the present, new features can still be discovered and explored. Technology is of great help for this and the newly developed, and still under development, tools for Automated Reasoning [16, 19] are central for the exploration and discovery of new theorems. They are based at a great extent on the theory of Gröbner bases and its various developments, on elimination ideals and resultants; see [5].

Many results have been conjectured using a Dynamic Geometry System (DGS), and proofs have been based on work assisted by a Computer Algebra System (CAS). It happens that a DGS includes a CAS; this is the case with GeoGebra, in which Giac is embedded [17]. Nevertheless, the discussion between DGS and CAS is still a dialog between different technologies [13, 20]. Specific developments have appeared where the dynamics of a DGS provide a strong visualization of the geometric situation, when the CAS provides a set of static snapshots, together with exact equations.

Another contribution of the CAS is via its package for polynomial rings. Sometimes, the automated commands of a DGS for the determination of geometric loci and envelopes provide a curve and a polynomial equation. The straightforward way to have a proof of the reducibility or irreducibility of the obtained curve is only by symbolic factorization. Hence the utility of having a CAS at hand.

A well known family of planar algebraic curves is defined by the so called Cassini ovals, which can be described in several ways, as we do so in what follows. However, the present work is devoted to the exploration of offsets of Cassini ovals by using several technological tools that surely the reader is aware of. Our explorative investigation provides a kind of exhaustive description of the offsets under study.

2 Cassini ovals

A *Cassini oval* is a plane curve $\mathcal{C}$ defined as follows.

Definition 1 *Take two distinct points F_1 and F_2 in the plane and a positive real b. Denote $a = F_1F_2$. The geometric locus of points M in the plane such that $MF_1 \cdot MF_2 = b^2$, if it is not empty, is called a Cassini oval.*

The points F_1 and F_2 are called the foci *of $\mathcal{C}$ and the number $e = b/a$ is called the* eccentricity *of the Cassini oval $\mathcal{C}$..*

Easy algebraic manipulations yield an implicit equation for $\mathcal{C}$.

Take $F_1(a, 0)$ and $F_2(-a, 0)$ for a given $a > 0$, and a generic point $M(x, y)$. Then the condition given in Def. 1 writes:

$$\sqrt{(x-a)^2 + y^2} \cdot \sqrt{(x+a)^2 + y^2} = b^2.$$

Squaring both sides, we obtain:

$$a^4 - 2a^2x^2 + 2a^2y^2 + x^4 + 2x^2y^2 + y^4 = b^4,$$

and this equation can be written as follows:

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 - b^4 = 0, \tag{1}$$

showing that a Cassini oval is a *quartic*.

Figure 1 shows examples, for fixed foci ($a = 3$) and various values of b.

The similarity with ellipses is not a coincidence. Giovanni Domenico Cassini (1625-1712) was an Italian born and French naturalized mathematician, engineer and astronomer. He discovered four satellites of the giant planet Saturn and thought that these ovals provided an accurate model for their orbits. A European probe called Cassini has been launched in 1997; it was the fourth probe to visit Saturn and the first to orbit this planet for years. It launched the lander Huygens to Titan, one of the most remarkable satellites of Saturn. Figure 2 shows a photo shot by the Cassini probe; the main gap within Saturn's rings system is also named after Cassini.

After Kepler's model for planetary motion was finally accepted, late in the 17^{th} century, it is well known that planets describe elliptic orbits around the Sun, with the Sun at one of the foci. The same model is valid for satellites (either natural or artificial) orbiting a planet. In general, the problem of finding the trajectory of an object under the action of a central force (such as gravitational attraction), and obeying Newton's Second Law of movement, is known as the Kepler's problem. The mathematical model for this problem is provided by a second order differential equation whose solutions describe the trajectory of the object. It turns out that these solutions are conic sections, and if they are closed, then the orbit of the object is an ellipse with one of the foci located at the punctual mass, which bears the central force.