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Locus Surfaces and Linear Transformations when Fixed Point is at an Infinity

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Abstract

We extend the locus problems discussed in [9], [10] and [12], for a quadric surface when the fixed point is at an infinity. This paper will benefit those students who have backgrounds in Linear Algebra and Multivariable Calculus. As we shall see that the transformation from a quadric surface $\sum$ to its locus surface Δ is a linear transformation. Consequently, how the eigenvectors are related to the position of the fixed point at an infinity will be discussed.

1 Introduction

In *[10]*, we consider the following:

Original problem: *We are given a fixed point A and a generic point C on a surface Σ. We let the line l pass through A and C and intersect a well-defined D on Σ, we want to determine the locus surface generated by the point E, lying on CD, which satisfies $\overrightarrow{ED} = s\overrightarrow{CD}$, where s is a real number parameter.*

We call the point D to be the antipodal point of C, and we often write the locus point as $E = sC+(1-s)D$ in our discussions with no confusions. We provide proofs in this paper, where the discussions originated from *[12]*, how the locus surface for a quadric shall behave when the fixed point $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$ is at an infinity; we remark that $\rho \to \infty$ and the point A depends on the angles u_0 and v_0. We recall from *[12]* that the locus surfaces, when the surface Σ is an ellipsoid or an hyperboloid with two sheets, we have found the exact expressions for the antipodal point D_{inf} corresponding to point C on Σ when A is at an infinity. In Sections 2 and 3, we discuss how the locus problems for an ellipsoid or a hyperboloid with two sheets can be described as a linear transformation and how their respective eigenvectors and eigenvalues are related to the behaviors of the corresponding locus surfaces. In Section 4, we give a geometric descriptions for the locus surfaces when the parameter s is a large number, including when $s \to \infty$.

2 The Locus Surface When Fixed Point is at an Infinity

If Σ is the quadric surface $F(x,y,z)=0$ we recall from *[10]* how we find the locus surface of Σ with respect to the fixed point $A=(x_0,y_0,z_0)$. We represent a generic point on Σ as

$$C=\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}. \tag{1}$$

We used Vieta's formulas to calculate the coordinates of point D, denoted by (x_1,y_1,z_1), which is the antipodal point of C and is the intersection between the quadric Σ and the line l passing through A and C. The point $E=sC+(1-s)D$, which is denoted by (x_e,y_e,z_e), generates the locus surface that we will explore in this paper. We remark that once the fixed point A is chosen, since A and C together determine the point E, the locus surface is thus fixed too. We write the locus surface as follows:

$$\Delta_A(C)=\begin{bmatrix}x_e\\ y_e\\ z_e\end{bmatrix}=\begin{bmatrix}s\hat{x}+(1-s)x_1\\ s\hat{y}+(1-s)y_1\\ s\hat{z}+(1-s)z_1\end{bmatrix}.$$

Unless otherwise specified in this paper, we focus on the parameter $s>1$ in this paper. In what follows, we shall simplify use Δ for a locus surface with no confusion.

We summarize from [12] how we find the locus of Σ with respect to a fixed point A, which is at an infinity.

1. Let the spherical coordinate for the fixed point A be $(\rho\cos u_0\sin v_0, \rho\sin u_0\sin v_0, \rho\cos v_0)$. If we define two auxiliary functions, namely

$$k\doteq k(\hat{x},\hat{y})=\frac{\hat{y}-y_0}{\hat{x}-x_0},\ \text{and} \tag{2}$$

$$m\doteq m(\hat{x},\hat{z})=\frac{\hat{z}-z_0}{\hat{x}-x_0} \tag{3}$$

2. We follow the usual procedure to find the intersection between the line AC and the quadric surface at $D=(x_1,y_1,z_1)$ respectively by adopting the Vieta's formula.

3. Next we let $\rho\to\infty$ to obtain the corresponding intersection point $D_{\inf}=(x_{1\inf},y_{1\inf},z_{1\inf})$.

4. The corresponding locus surface, is defined as $E_{\inf}=(x_{e\inf},y_{e\inf},z_{e\inf})$ where

$$\begin{aligned}x_{e\inf}&=s\hat{x}+(1-s)\left(x_{1\inf}\right)\\ y_{e\inf}&=s\hat{y}+(1-s)\left(y_{1\inf}\right)\\ z_{e\inf}&=s\hat{z}+(1-s)\left(z_{1\inf}\right).\end{aligned}$$

If $A=(\rho\cos u_0\sin v_0,\rho\sin u_0\sin v_0,\rho\cos v_0)$. Let us note that (2) and (3) become,

$$k=\frac{\hat{y}-\rho\sin u_0\sin v_0}{\hat{x}-\rho\cos u_0\sin v_0},\ \text{and} \tag{4}$$

$$m=\frac{\hat{z}-\rho\cos v_0}{\hat{x}-\rho\cos u_0\sin v_0}. \tag{5}$$