Numer. Math. Theor. Meth. Appl. Vol. **15**, No. 4, pp. 1193-1218
doi: 10.4208/nmtma.OA-2022-0005s November 2022

Approximation of the Spectral Fractional Powers of the Laplace-Beltrami Operator

Andrea Bonito[1,*] and Wenyu Lei[2]

[1] *Department of Mathematics, Texas A&M University, College Station, TX 77843, USA*
[2] *SISSA - International School for Advanced Studies, Via Bonomea 265, 34136 Trieste, Italy*

Received 13 January 2021; Accepted (in revised version) 6 February 2022

Abstract. We consider numerical approximation of spectral fractional Laplace-Beltrami problems on closed surfaces. The proposed numerical algorithms rely on their Balakrishnan integral representation and consists a sinc quadrature coupled with standard finite element methods for parametric surfaces. Possibly up to a log term, optimal rate of convergence are observed and derived analytically when the discrepancies between the exact solution and its numerical approximations are measured in L^2 and H^1. The performances of the algorithms are illustrated on different settings including the approximation of Gaussian fields on surfaces.

AMS subject classifications: 65M12, 65M15, 65M60, 35S11, 65R20

Key words: Fractional diffusion, Laplace-Beltrami, FEM parametric methods on surfaces, Gaussian fields.

1. Introduction

In this paper, we consider finite element approximations of the spectral fractional Laplace-Beltrami problem

$$(-\Delta_\gamma)^s \widetilde{u} = \widetilde{f} \quad \text{on } \gamma, \tag{1.1}$$

where γ is a closed, compact and C^3 hyper-surface in $\mathbb{R}^n$, $n = 2, 3$ and where the data f is square integrable in γ while satisfying the compatibility condition $\int_\gamma \widetilde{f} = 0$. Such system finds applications for instance in the study of Gaussian random fields on surfaces [5, 26, 27].

As we shall see, the spectral negative fractional powers of the Laplace-Beltrami operator $L := -\Delta_\gamma$ are defined using the Balakrishnan formula [3]

*Corresponding author. *Email addresses:* `wenyu.lei@sissa.it` (W. Lei), `bonito@math.tamu.edu` (A. Bonito)

$$L^{-s}\widetilde{f} := \frac{\sin(\pi s)}{\pi}\int_0^\infty \mu^{-s}(\mu I + L)^{-1}\widetilde{f}\, d\mu, \quad s \in (0,1). \tag{1.2}$$

This representation is equivalent to the spectral representation more often used as detailed in Section 2.3.

On Euclidean domains, a two-step numerical approximation strategy based on (1.2) is proposed in [13, 14] for the standard Laplacian. After a change of variable, an exponentially converging sinc quadrature [11]

$$\mathcal{Q}_k^{-s}(L)\widetilde{f} = \frac{k\sin(\pi s)}{\pi}\sum_{\ell=-M}^{N} e^{(1-s)y_\ell}(e^{y_\ell}I + L)^{-1}\widetilde{f} \tag{1.3}$$

is advocated for the approximation of the outer integral in (1.2). Here $k > 0$ is the quadrature spacing, $y_\ell = \ell k$, and $M, N \in \mathcal{O}(k^{-2})$ are positive integers. In [13,14], each of the sub-problems $(e^{y_\ell}I + L)^{-1}\widetilde{f}$ are approximated independently using a standard continuous piecewise linear finite element method. We refer to [4] for extensions to hp-discretization methods. For completeness, we also mention the reviews [6, 28] describing alternate algorithms for fractional powers of elliptic operators on Euclidean domains.

Regarding the Laplace-Beltrami operator on closed surface, several algorithms for its approximation are available as well. Parametric finite element methods [20] and trace finite element methods [17, 30, 31] rely on polygonal approximations Γ of the exact surface γ. Instead, other numerical methods like the narrow band methods [16, 17] are defined on a neighborhood of γ. We refer to [10, 21] for reviews of these methods and others. All the above mentioned numerical approaches have in common an inherent geometric errors (or consistency errors) resulting from the approximation of γ, thereby leading to the uncharted territory (in the context of approximation of fractional operators) of non-conforming methods.

The main contribution of this work is to develop a new analysis to incorporate the approximation by parametric finite element methods within the framework developed in [14]. The approximate polygonal surface on which the finite element method is defined is related to the exact surface via a C^2 orthogonal projection $\mathbf{P} : \Gamma \to \gamma$ given by $\mathbf{P}(\mathbf{x}) = \mathbf{x} - d(\mathbf{x})\nabla d(\mathbf{x})$, where d is the signed distance function to γ. In [19, 20] this typical choice of lift is put forward in view of its $\mathcal{O}(h^2)$ approximation of the geometry at the expense of requiring surfaces γ of class C^3. Here h stands for typical diameter of the faces constituting the approximation Γ. In opposition, favoring practical implementations and applicability to rougher surfaces, a generic continuous piecewise C^2 lift is considered in [7,8,29] leading to a reduced $\mathcal{O}(h)$ approximation of the geometry. The latter is sufficient to control with optimal order the H^1, but not L^2, discrepancy between the solution to the standard Laplace-Beltrami operator and its parametric linear finite element approximations. However, the analysis of algorithms based on the integral representation (1.2) requires space discretization methods to perform simultaneously well in H^1 and L^2, thereby exacerbating even more the intricate relation