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A Survey of the L1 Scheme in the Discretisation of Time-Fractional Problems

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Abstract. A survey is given of convergence results that have been proved when the L1 scheme is used to approximate the Caputo time derivative D_t^α (where $0 < \alpha < 1$) in initial-boundary value problems governed by $D_t^\alpha u - \Delta u = f$ and similar equations, while taking into account the weak singularity that is present in typical solutions of such problems. Various aspects of these analyses are outlined, such as global and local convergence bounds and the techniques used to derive them, fast implementation of the L1 scheme, semilinear problems, multi-term time derivatives, α-robustness, a posteriori error analysis, and two modified L1 schemes that achieve better accuracy. Over fifty references are provided in the bibliography, more than half of which are from the period 2019-2022.

AMS subject classifications: 65M10, 78A48
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1. Introduction

The L1 finite difference scheme is the simplest and most popular method for discretising Caputo fractional derivatives of order $\alpha \in (0,1)$. In this survey we aim to give an overview of its use and analysis in problems governed by differential equations that include a Caputo time derivative.

The L1 scheme first appeared in [45, Section 8.2]. The terminology "L1" has no connection with the well-known Lebesgue space of the same name; instead, the "1" refers to the numerical approximation of Caputo derivatives of order $\alpha \in (0,1)$ by means of a piecewise linear form of product integration (likewise, in [45] there is an L2 formula for approximating Caputo derivatives of order $\alpha \in (1,2)$ based on piecewise quadratic product integration, and one can analogously construct formulas L3, L4, etc. for higher-order Caputo derivatives), while the "L" is perhaps by association with Riemann-Liouville (in [45, Section 8.2] one also finds "R" formulas for approximating fractional integrals).

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The paper is structured as follows. The initial-boundary value problem governed by $D_t^\alpha u - \Delta u + cu = f$ (here $D_t^\alpha u$ is a Caputo derivative of order α with $0 < \alpha < 1$) that is our main focus is described in Section 2, where we emphasise the weak singularity at the initial time $t = 0$ that appears in its typical solutions. Then the L1 scheme is presented in Section 3, together with a brief synopsis of its fast implementation. Section 4 is the heart of our paper, where global and local convergence estimates for the L1 scheme are presented and compared. Here we draw the reader's attention to Remark 4.5, which highlights the best known error estimate for this scheme (when one has sufficient regularity and compatibility of the data). Variations of the original differential equation are considered in Section 5, where nonlinear (mainly semilinear) differential equations are discussed, and Section 6, where $D_t^\alpha u$ is replaced by a sum of Caputo derivatives. In Section 7 α-robust analyses of numerical methods are investigated. Two modifications of the L1 scheme, which enhance its accuracy, appear in Section 8. A significant development in the a posteriori analysis of the L1 scheme is outlined in Section 9. Finally, in Section 10 a short list of papers suitable for learning specific analytical techniques is given.

Remark 1.1. Over 150 published papers have used the L1 scheme to approximate a Caputo derivative; to discuss all of them would make this survey far too long. Thus, our bibliography refers only to a representative sample of L1 papers that cover the main developments in the analysis and use of this scheme; the reader perusing these papers will find in them many references to other L1 papers.

Remark 1.2. Typical solutions of the problem that we study exhibit a weak singularity at the initial time $t = 0$; see Section 2. Our survey ignores several papers that use the L1 scheme but assume the unknown solution is smooth at $t = 0$, as this assumption makes the error analysis much easier but is rarely satisfied (except for artificially-constructed problems where one begins by choosing a smooth solution then one fits the right-hand side of the differential equation to it).

Notation and errors displayed. We consider initial-boundary value problems in space and time, and consequently there are both spatial and temporal components in the error of the solution computed by any numerical method, but in our discussions of error we shall describe only the temporal component (i.e., the spatial component of error is omitted) since our focus is on the L1 scheme for discretising the fractional time derivative. Throughout the paper C is a generic constant that depends on the data of the problem but is independent of the temporal mesh parameter N and of the spatial mesh diameter.

2. The time-fractional diffusion problem

The L1 scheme appears most often in the literature when solving time-fractional diffusion problems of the type that we now describe, together with their variants and extensions.