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The Variable-Step L1 Scheme Preserving a Compatible Energy Law for Time-Fractional Allen-Cahn Equation

Hong-Lin Liao[1,2], Xiaohan Zhu[2] and Jindi Wang[3,*]

[1] *MIIT Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles, Nanjing 211106, P.R. China*
[2] *School of Mathematics, Nanjing University of Aeronautics and Astronautics, 211106, P.R. China*
[3] *School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, P.R. China*

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Abstract. In this work, we revisit the adaptive L1 time-stepping scheme for solving the time-fractional Allen-Cahn equation in the Caputo's form. The L1 implicit scheme is shown to preserve a variational energy dissipation law on arbitrary nonuniform time meshes by using the recent discrete analysis tools, i.e., the discrete orthogonal convolution kernels and discrete complementary convolution kernels. Then the discrete embedding techniques and the fractional Grönwall inequality are applied to establish an L^2 norm error estimate on nonuniform time meshes. An adaptive time-stepping strategy according to the dynamical feature of the system is presented to capture the multi-scale behaviors and to improve the computational performance.

AMS subject classifications: 35Q99, 65M06, 65M12, 74A50

Key words: Time-fractional Allen-Cahn equation, adaptive L1 scheme, variational energy dissipation law, orthogonal convolution kernels, complementary convolution kernels.

1. Introduction

We consider the numerical approximations for time-fractional Allen-Cahn (TFAC) equation

$$\partial_t^\alpha \Phi = -\kappa\mu, \quad \text{where the potential} \quad \mu := f(\Phi) - \epsilon^2\Delta\Phi \tag{1.1}$$

on a bounded regular domain $\mathbf{x} \in \Omega \subseteq \mathbb{R}^2$ subject to periodic boundary conditions.

*Corresponding author. *Email addresses:* liaohl@csrc.ac.cn, liaohl@nuaa.edu.cn (H.-L. Liao), wangjindixy@163.com (J. Wang), cyzhuxiaohan@163.com (X. Zhu)

Here, $\epsilon > 0$ is an interface width parameter, $\kappa > 0$ is the mobility coefficient, and the nonlinear bulk force $f(\Phi)$ is taken as the polynomial double-well potential $f(\Phi) = \Phi^3 - \Phi$. The notation $\partial_t^\alpha := {}_0^C D_t^\alpha$ in (1.1) represents the fractional Caputo derivative of order α with respect to t, that is,

$$\left(\partial_t^\alpha v\right)(t) := \left(\mathcal{I}_t^{1-\alpha} v'\right)(t), \quad 0 < \alpha < 1, \tag{1.2}$$

in which the fractional Riemann-Liouville integral $\mathcal{I}_t^\beta$ of order $\beta > 0$ is given by

$$\left(\mathcal{I}_t^\beta v\right)(t) := \int_0^t \omega_\beta(t-s) v(s)\,\mathrm{d}s, \tag{1.3}$$

where

$$\omega_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}.$$

As well known, the energy dissipation law is an important and essential property of the classical phase field models. Recall the following Ginzburg-Landau energy functional [1]:

$$E[\Phi] := \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla \Phi|^2 + F(\Phi) \right) \mathrm{d}\mathbf{x}, \tag{1.4}$$

where

$$F(\Phi) = \frac{1}{4}\left(\Phi^2 - 1\right)^2.$$

The classical Allen-Cahn (AC) model preserves the following energy dissipation law:

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \kappa \left\| \frac{\delta E}{\delta \Phi} \right\|^2 = 0, \quad t > 0, \tag{1.5}$$

where the inner product

$$(u, v) := \int_\Omega uv\,\mathrm{d}\mathbf{x},$$

and the associated L^2 norm $\|u\| := \sqrt{(u,u)}$ for all $u, v \in L^2(\Omega)$. It is of great interest to design some numerical algorithms that preserve the energy dissipation law at each time level because non-energy-stable numerical schemes would not accurately capture the coarsening dynamics or lead to numerical instability.

For the classical gradient flows, there are several effective strategies to develop energy stable numerical algorithms, such as the convex splitting method [2,26], stabilization technique [23,27], invariant energy quadratization approach [5,6] and scalar auxiliary variable formulation [22]. Compared with the classical phase field models, however, the theoretical works regarding the energy stable property of the time-fractional phase field models are limited. It was shown [25, Theorem 4.2] that the TFAC model (1.1) admits the maximum bound principle

$$|\Phi(\mathbf{x}, t)| \le 1, \quad \text{if} \quad |\Phi(\mathbf{x}, 0)| \le 1, \quad t > 0, \tag{1.6}$$