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Galerkin Finite Element Approximation for Semilinear Stochastic Time-Tempered Fractional Wave Equations with Multiplicative Gaussian Noise and Additive Fractional Gaussian Noise

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Abstract. To model wave propagation in inhomogeneous media with frequency dependent power-law attenuation, it is needed to use the fractional powers of symmetric coercive elliptic operators in space and the Caputo tempered fractional derivative in time. The model studied in this paper is semilinear stochastic space-time fractional wave equations driven by infinite dimensional multiplicative Gaussian noise and additive fractional Gaussian noise, because of the potential fluctuations of the external sources. The purpose of this work is to discuss the Galerkin finite element approximation for the semilinear stochastic fractional wave equation. First, the space-time multiplicative Gaussian noise and additive fractional Gaussian noise are discretized, which results in a regularized stochastic fractional wave equation while introducing a modeling error in the mean-square sense. We further present a complete regularity theory for the regularized equation. A standard finite element approximation is used for the spatial operator, and a mean-square priori estimates for the modeling error and the approximation error to the solution of the regularized problem are established. Finally, numerical experiments are performed to confirm the theoretical analysis.

AMS subject classifications: 35R11, 60H15, 65M12, 65M60, 60G22

Key words: Galerkin finite element method, semilinear stochastic time-tempered fractional wave equation, fractional Laplacian, multiplicative Gaussian noise, additive fractional Gaussian noise.

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1. Introduction

The classical wave equation well models the wave propagation in an ideal medium. However, the wave propagation in complex inhomogeneous media generally has frequency-dependent attenuation, being observed in a wide range of areas including acoustics, viscous damping in the seismic isolation of buildings, structural vibration, and seismic wave propagation [7, 21, 24, 31]. The striking power-law feature of the attenuated wave propagation implies that the Laplacian in the classical equation should be replaced by fractional powers of symmetric coercive elliptic operators in space, while the time-tempered derivative should be substituted for second time derivative. Because of the finite time/space scale, the tempered power-law distribution in some sense becomes more reasonable choice compared with the pure power-law one [23]. There are already some discussions on the numerical methods or correct ways of specifying the boundary conditions for tempered fractional differential equations; see, e.g., [11, 12, 15, 34] and the references therein. As for the fractional wave equations, there are also some progresses not only on their numerical methods [9, 14, 30, 33] but also on their fundamental solutions and properties [4, 17].

Random effects arise naturally in practically physical systems; the ones considered in this paper are on the fluctuations of the external sources, and the fluctuations include both infinite dimensional multiplicative Gaussian noise and additive fractional Gaussian noise, which drive the semilinear space-time fractional wave equations. The multiplicative noise can capture the effects of geometrical confinements [20]. The fractional Gaussian noise is the formal derivative of the fractional Brownian motion (FBM) B^H, being a centered Gaussian process with a special covariance function determined by Hurst parameter $H \in (0,1)$. For $H = 1/2$, $B^{1/2}$ is the standard Brownian motion, the formal time derivative of which is white noise. For $H \neq 1/2$, B^H behaves in a way completely different from the standard Brownian motion; especially, neither is a semi-martingale nor a Markov process. In addition, the FBM with Hurst parameter $H \in (1/2,1)$ enjoys the property of a long range memory, which roughly implies that the decay of stochastic dependence with respect to the past is only sub-exponentially slow. This long-range dependence property of the FBM makes it a realistic choice of noise for problems with long memory in the applied sciences.

With the above introduction of the fractional wave equation and the external noises, now we propose the model, which is a space-time fractional wave equation driven by three nonlinear external source terms: a deterministic term and two stochastic terms, being respectively Gaussian noise and fractional Gaussian noise. Specifically, the model is a semilinear stochastic time tempered fractional wave equation with $3/2 < \alpha < 2$, $1/2 < \beta < 1$, $1/2 < H < 1$, and $\nu > 0$

$$\begin{aligned} &{}_0^c\partial_t^{\alpha,\nu} u(t,x) + (-\Delta)^\beta u(t,x) \\ &= f\big(t,u(t,x)\big) + g\big(t,u(t,x)\big)\frac{\partial^2 \mathbb{W}(t,x)}{\partial t \partial x} + h(t)\frac{\partial^2 \mathbb{W}^H(t,x)}{\partial t \partial x} \quad \text{in } (0,T] \times \mathcal{D}, \end{aligned} \tag{1.1a}$$