Commun. Math. Res.
doi: 10.4208/cmr.2021-0101

Vol. **39**, No. 1, pp. 1-35
February 2023

Local Dispersive and Strichartz Estimates for the Schrödinger Operator on the Heisenberg Group

Hajer Bahouri[1,*] and Isabelle Gallagher[2,3]

[1] *CNRS & Sorbonne Université Laboratoire Jacques-Louis Lions (LJLL) UMR 7598 4, Place Jussieu 75005 Paris, France.*
[2] *DMA, École Normale Supérieure, CNRS, PSL Research University, 75005 Paris, France.*
[3] *UFR de Mathématiques, Université de Paris, 75013 Paris, France.*

Received 4 December 2021; Accepted 17 March 2022

Abstract. It was proved by Bahouri *et al.* [9] that the Schrödinger equation on the Heisenberg group $\mathbb{H}^d$, involving the sublaplacian, is an example of a totally non-dispersive evolution equation: for this reason global dispersive estimates cannot hold. This paper aims at establishing local dispersive estimates on $\mathbb{H}^d$ for the linear Schrödinger equation, by a refined study of the Schrödinger kernel S_t on $\mathbb{H}^d$. The sharpness of these estimates is discussed through several examples. Our approach, based on the explicit formula of the heat kernel on $\mathbb{H}^d$ derived by Gaveau [19], is achieved by combining complex analysis and Fourier-Heisenberg tools. As a by-product of our results we establish local Strichartz estimates and prove that the kernel S_t concentrates on quantized horizontal hyperplanes of $\mathbb{H}^d$.

AMS subject classifications: 43A30, 43A80

Key words: Heisenberg group, Schrödinger equation, dispersive estimates, Strichartz estimates.

---

*Corresponding author. *Email addresses:* hajer.bahouri@sorbonne-universite.fr (H. Bahouri), gallagher@math.ens.fr (I. Gallagher)

1 Introduction

1.1 Setting of the problem

It is well-known that the solution to the free Schrödinger equation on $\mathbb{R}^n$

$$(S) \quad \begin{cases} i\partial_t u - \Delta u = 0, \\ u_{|t=0} = u_0 \end{cases}$$

can be explicitly written with a convolution kernel for $t \neq 0$

$$u(t,\cdot) = u_0 \star \frac{\mathrm{e}^{-i\frac{|\cdot|^2}{4t}}}{(-4\pi i t)^{\frac{n}{2}}} \cdot \tag{1.1}$$

The proof of this explicit representation stems by a combination of Fourier and complex analysis arguments, from the expression of the heat kernel on $\mathbb{R}^n$. More precisely, taking the partial Fourier transform of (S) with respect to the variable x and integrating in time the resulting ODE, we get

$$\widehat{u}(t,\xi) = \mathrm{e}^{it|\xi|^2} \widehat{u}_0(\xi),$$

where for any function $g \in L^1(\mathbb{R}^n)$ we have defined

$$\widehat{g}(\xi) \stackrel{\text{def}}{=} \mathcal{F}(g)(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \mathrm{e}^{-i\langle x,\xi\rangle} g(x)\, dx.$$

The heart of the matter to prove (1.1) then consists in computing in the sense of distributions the inverse Fourier transform of the complex Gaussian

$$\left(\mathcal{F}^{-1} \mathrm{e}^{it|\cdot|^2}\right)(x) = \frac{\mathrm{e}^{-i\frac{|x|^2}{4t}}}{(-4\pi i t)^{\frac{n}{2}}} \cdot \tag{1.2}$$

The proof of formula (1.2) is based on two observations: first, that for any x in $\mathbb{R}^n$, the two maps

$$z \in \mathbb{C} \longmapsto H_1(z) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{i\langle x,\xi\rangle} \mathrm{e}^{-z|\xi|^2} d\xi,$$

$$z \in \mathbb{C} \longmapsto H_2(z) \stackrel{\text{def}}{=} \frac{1}{(4\pi z)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|x|^2}{4z}}$$