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On Instability of the Rayleigh–Bénard Problem without Thermal Diffusion in a Bounded Domain under L^1-Norm

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Abstract. We investigate the thermal instability of a three-dimensional Rayleigh–Bénard (RB for short) problem without thermal diffusion in a bounded domain. First we construct unstable solutions in exponential growth modes for the linear RB problem. Then we derive energy estimates for the nonlinear solutions by a method of a prior energy estimates, and establish a Gronwall-type energy inequality for the nonlinear solutions. Finally, we estimate for the error of L^1-norm between the both solutions of the linear and nonlinear problems, and prove the existence of escape times of nonlinear solutions. Thus we get the instability of nonlinear solutions under L^1-norm.

AMS subject classifications: 76E06, 76D05

Key words: Rayleigh–Bénard problem, thermal instability, initial-boundary value problem.

1 Introduction

Thermal instability often arises when a fluid is heated from below. The phenomenon of thermal convection itself had been recognized by Rumford [24] and Thomson [25]. However, the first quantitative experiment on thermal instability and the recognition

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of the role of viscosity in the phenomenon are due to Bénard [2]. The Bénard convection can be modeled by the (nonlinear) compressible Navier–Stokes–Fourier (simplified by NSF) equations [11]. Since the continuity equation in the NSF equations is hyperbolic, thus it is difficult to theoretically investigate the thermal convection. Later Rayleigh investigated thermal convection by using Boussinesq (approximation) equations, in which the density is considered as a constant in all the terms of the equations except for the gravity term that is assumed to vary linearly with the temperature [4]. Compared with the compressible NSF model, the Boussinesq model is fully parabolic due to the absence of the continuity equation. Thus, based on the linearized Boussinesq model, Rayleigh first theoretically provided the instability criterion for the occurrence of thermal convection [3, 23]. After Rayleigh's pioneering work, the instability criterion had been further mathematically verified for the nonlinear Boussinesq model in the Hadamard sense by the energy method and the bootstrap instability method, see [13, 20] for examples. At present, it has been also widely investigated how the thermal instability evolves under the effects of other physical factors, such as the elasticity [19], rotation [7, 10], the magnetic field [6, 8, 9], surface tension [22] and so on. Recently Ma and Wang also established mathematical theory of attractor bifurcation for two-dimensional Boussinesq model [21]. However, the corresponding three-dimensional case is still an open problem.

It is physically well-known that the system of nonlinear Boussinesq equations is always unstable, if the thermal diffusion is absent. However there is not any available mathematical proof for this physical assertion. In this paper, we try to mathematically prove this assertion. To begin with, we shall introduce the three-dimensional (3D for short) Rayleigh–Bénard (RB for short) equations without thermal diffusion in a bounded domain Ω:

$$\begin{cases} v_t+v\cdot\nabla v+\nabla p/\rho=g(\alpha(\Theta-\Theta_{\rm b}-1)e_3)+\mu\Delta v, \\ \Theta_t+v\cdot\nabla\Theta=0, \\ {\rm div} v=0. \end{cases} \tag{1.1}$$

Next we further explain the notations in the equations above.

The unknowns $v=v(x,t)$, $\Theta=\Theta(x,t)$ and $p=p(x,t)$ denote velocity, temperature and pressure of an incompressible fluid, resp.. The parameters $\rho,\alpha,\mu>0$ and $g>0$ denote the density constant at some properly chosen temperature parameter $\Theta_{\rm b}$, the coefficient of volume expansion, shear viscosity coefficient, and the gravitational constant, respectively. $g\rho\alpha(\Theta-\Theta_{\rm b})e_3$ stands for the buoyancy, $-\rho g e_3$ for the gravitational force, where $e_3=(0,0,1)^{\rm T}$ and T denotes the transposition.

The rest state of the system (1.1) can be given by $r_{\rm B}:=(0,\bar{\Theta})$ with an associated pressure profile $\bar{p}$, where the temperature profile $\bar{\Theta}$ and $\bar{p}$ depend on x_3 only, and