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Dual Control Methods for a Mixed Control Problem with Optimal Stopping Arising in Optimal Consumption and Investment

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Abstract. This paper studies a problem of optimal investment and consumption with early retirement option under constant elasticity variation (CEV) model with finite horizon. Two risky assets are involved in the model with one following geometric Brownian motion and the other a CEV model. This problem is a kind of two dimensional mixed control and optimal stopping problems with finite horizon. The existence and continuity of the optimal retirement threshold surfaces are proved and the working and retirement regions are characterized theoretically. Least-squares Monte-Carlo methods are developed to solve this mixed control and optimal stopping problem. The algorithms are well implemented and the optimal retirement threshold surfaces, optimal investment strategies and the optimal consumptions are drawn via examples.

AMS subject classifications: 49M25, 91B70, 65C05

Key words: Optimal investment and consumption, stochastic control with optimal stopping, nonlinear free boundary problems, least-squares Monte-Carlo methods.

1. Introduction

In this paper, we focus on a problem of optimal consumption and investment with mandatory retirement date and early retirement option. The agent must retire at the mandatory retirement date and has an option to retire earlier than the mandatory retirement date. This kind of problem can be reduced to a mixed control and optimal stopping problem with finite horizon. If only one feature of retirement, either voluntary retirement or mandatory retirement, is involved, then the problem of optimal

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consumption and investment is relatively easier to solve. With only voluntary retirement being involved, the problem can be cast into an infinite horizon mixed control problem and it can be solved more-or-less analytically. For instance, Bodie *et al.* [1] examine consumption and investment decisions in a life-cycle model where retirement is taken into account by specifying an age at which labor earnings stop, but consumption spending continues. Choi *et al.* [2] study an investor's decision to switch from active portfolio management to passive management. Choi and Shim [3] study the optimal retirement and consumption/investment choice of an infinitely-lived economic agent and have solved for the optimal choice of retirement time. Choi *et al.* [4], Farhi and Panageas [7] investigated optimal portfolio, consumption-leisure and retirement choice of an infinitely-lived economic agent and solved the problem analytically. Dybvig and Liu [5, 6] study the lifetime consumption and investment with voluntary retirement and constrained borrowing. With only feature of mandatory retirement, the optimal retirement and consumption/investment is just a kind of optimal control without discretionary stopping.

If the mandatory retirement date and early retirement option are considered together, then the optimal consumption and investment become a kind of a mixed control and optimal stopping problem with finite horizon. The mixed control and stopping problem can be dated back to the research by Karatzas and Wang [9] in which the dual control method is applied. But the methods cannot be directly used to the problem of optimal consumption and investment with mandatory retirement date and early retirement option as the state equations before and after the retirement date are different and the dual control problems in the two stages are not the same. Yang and Koo [12] first investigate the problem of consumption and investment with both mandatory retirement date and early retirement option. They analyze the properties of the optimal strategy and provide a characterization of the threshold of wealth as a function of time. However that paper does not study the solution of the problem as it has no analytical solutions and is very hard to solve numerically.

In this paper, we study the problem of optimal investment and consumption with early retirement option under constant elasticity variation (CEV) model with finite horizon, which can be regarded as the extension of the work by Yang and Koo [12] from one dimension to two dimension. In fact, a geometric Brownian motion (GBM) can be regarded as a special of a CEV process when its elasticity β equals to 0, which represents a kind of assets with fixed volatility. CEV process represents those assets with stochastic volatility. As a result, we use the combined setting of assets as a better representation of the financial market, compared to [12]. The existence and property of the optimal retirement threshold are proved and the working and retirement regions are characterized theoretically. Compared to [12], the optimal retirement threshold is not a curve of wealth in time, instead it is a surface of wealth and stock price in time. In addition, the analysis is obviously more complicated than that of Yang and Koo [12]. More distinct contribution of this paper is that the least-squares Monte-Carlo methods are derived to solve the mixed control and optimal stopping problems. The least-squares Monte-Carlo methods have been widely used in the American option pricing (see e.g., [10]).