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Spectral Analysis for Preconditioning of Multi-Dimensional Riesz Fractional Diffusion Equations

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Abstract. In this paper, we analyze the spectra of the preconditioned matrices arising from discretized multi-dimensional Riesz spatial fractional diffusion equations. The finite difference method is employed to approximate the multi-dimensional Riesz fractional derivatives, which generates symmetric positive definite ill-conditioned multi-level Toeplitz matrices. The preconditioned conjugate gradient method with a preconditioner based on the sine transform is employed to solve the resulting linear system. Theoretically, we prove that the spectra of the preconditioned matrices are uniformly bounded in the open interval $(\frac{1}{2}, \frac{3}{2})$ and thus the preconditioned conjugate gradient method converges linearly within an iteration number independent of the discretization step-size. Moreover, the proposed method can be extended to handle ill-conditioned multi-level Toeplitz matrices whose blocks are generated by functions with zeros of fractional order. Our theoretical results fill in a vacancy in the literature. Numerical examples are presented to show the convergence performance of the proposed preconditioner that is better than other preconditioners.

AMS subject classifications: 65F08, 65M10, 65N99

Key words: Multi-dimensional Riesz fractional derivative, multi-level Toeplitz matrix, sine transform based preconditioner, preconditioned conjugate gradient method.

1. Introduction

In this paper, we study a preconditioning technique for the following multi-dimensional Riesz fractional diffusion equations:

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$$-\sum_{i=1}^{m} d_i \frac{\partial^{\alpha_i} u(\mathbf{x})}{\partial |x_i|^{\alpha_i}} = y(\mathbf{x}), \quad \mathbf{x} \in \Omega = \prod_{i=1}^{m} [a_i, b_i] \subset \mathbb{R}^m, \tag{1.1}$$

subject to the boundary condition

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega,$$

where $d_i > 0$ for $i = 1, \ldots, m, \mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m, y(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$ is the source term, and $\frac{\partial^{\alpha_i} u(\mathbf{x})}{\partial |x_i|^{\alpha_i}}$ is the Riesz fractional derivative of order $\alpha_i \in (1,2)$ with respect to x_i defined by [32, 33]

$$\frac{\partial^{\alpha_i} u(\mathbf{x})}{\partial |x_i|^{\alpha_i}} = c(\alpha_i) \left({}_{a_i}D_{x_i}^{\alpha_i} u(\mathbf{x}) + {}_{x_i}D_{b_i}^{\alpha_i} u(\mathbf{x}) \right), \quad c(\alpha_i) = \frac{-1}{2\cos(\alpha_i \pi/2)} > 0. \tag{1.2}$$

The above left and right Riemann-Liouville (RL) fractional derivatives are defined by

$${}_{a_i}\mathrm{D}_{x_i}^{\alpha_i} u(\mathbf{x}) = \frac{1}{\Gamma(2-\alpha_i)} \frac{\partial^2}{\partial x_i^2} \int_{a_i}^{x_i} \frac{u(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_m)}{(x_i - \xi)^{\alpha_i - 1}} \mathrm{d}\xi, \tag{1.3}$$

$${}_{x_i}\mathrm{D}_{b_i}^{\alpha_i} u(\mathbf{x}) = \frac{1}{\Gamma(2-\alpha_i)} \frac{\partial^2}{\partial x_i^2} \int_{x_i}^{b_i} \frac{u(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_m)}{(\xi - x_i)^{\alpha_i - 1}} \mathrm{d}\xi, \tag{1.4}$$

respectively, where $\Gamma(\cdot)$ is the gamma function.

Fractional calculus has received an increasing interest since its applications involve various fields such as physics, chemistry, engineering, see [11,15,17,27]. The fractional diffusion equations, which are recognized as a class of important fractional differential equations, have been extensively applied in describing anomalous process and complicated phenomena such as long-range interactions [12], nonlocal dynamics [32], and so on.

Since the analytical solutions towards fractional differential equations are usually unaccessible, numerical schemes have been developed widely for solving the fractional differential equations, see [7, 16, 23, 33]. However, because of the nonlocal property of the fractional operators, the numerical discretization often gives rise to dense coefficient matrices, which yields that the direct solvers for the linear systems are time-consuming. Fortunately, the discretization matrices arising from fractional diffusion operators possess Toeplitz or Toeplitz-like structures, whose matrix-vector multiplication can be fast implemented by means of fast Fourier transform (FFT) [34]. Therefore, the iterative methods, such as the Krylov subspace method, are widely used to solve linear systems stemming from discretized fractional diffusion equations. Interestingly, it is found that such kind of Toeplitz matrix can be generated by a continuous real-valued even function with α-th order zeros at the origin $(1 < \alpha < 2)$. Accordingly, the condition number of the matrix is unbounded as the matrix size tends to infinity [6, 8, 29]. Therefore, the linear system arising from fractional diffusion equation is ill-conditioned, which leads to the iterative methods converging at a slower speed.