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MULTIPLE POSITIVE SOLUTIONS FOR A FOURTH-ORDER NONLINEAR EIGENVALUE PROBLEM*†

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Abstract

In this paper, by using the Guo-Krasnoselskii's fixed-point theorem, we establish the existence and multiplicity of positive solutions for a fourth-order nonlinear eigenvalue problem. The corresponding examples are also included to demonstrate the results we obtained.

Keywords positive solutions; eigenvalue problem; fixed point
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1 Introduction

In the past decades, an increasing interest in the existence and multiplicity of positive solutions for boundary value problems has been evolved by using some fixed-point theorems, for example, by the Krasnoselskii's fixed-point theorem, Ma [1] and Li [2] respectively established the existence and multiplicity of positive solutions for some fourth-order boundary value problems. Zhong [3] established the existence of at least one positive solution for the following four-point boundary value problem

$$\begin{cases} y^{(4)}(t) - f(t, y(t), y''(t)) = 0, \quad 0 \le t \le 1, \\ y(0) = y(1) = 0, \\ ay''(\xi_1) - by'''(\xi_1) = 0, \quad cy''(\xi_2) + dy'''(\xi_2) = 0. \end{cases}$$

In 2015, Wu [4] obtained some new results on the existence of at least one positive solution for the following fourth-order three-point nonlinear eigenvalue problem

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$$\begin{cases} u^{(4)}(t) = \lambda h(t) f(t, u(t)), \quad 0 \le t \le 1, \\ u(0) = u(1) = 0, \\ au''(\eta) - bu'''(\eta) = 0, \quad cu''(1) + du'''(1) = 0. \end{cases}$$

Bai [5] obtained the existence of triple positive solutions via the Leggett-Williams fixed-point theorem [6]. There are other meaningful investigated results on the existence of positive solutions for some types of nonlinear differential equations, one can be referred to [1-4,7,8]. But to the best of our knowledge, there are not many results on the existence of multiple positive solutions for fourth-order nonlinear eigenvalue problems with multi-points boundary value condition.

Based on the fact, our purpose in this paper is to investigate the existence and multiplicity of positive solutions for the following fourth-order three-point eigenvalue problem

$$\begin{cases} u^{(4)}(t) = \lambda h(t) f(t, u(t), u''(t)), \quad 0 \le t \le 1, \\ u(0) = u(1) = 0, \\ au''(\xi) - bu'''(\xi) = 0, \quad cu''(1) + du'''(1) = 0, \end{cases} \tag{1.1}$$

where λ is a positive parameter, $0<\frac{1}{4}<\xi<\frac{2}{3}<1$, a, b, c, d are nonnegative constants satisfying $ad+bc+ac>0$, $b-a\xi\ge 0$, $h(t)\in C[0,1]$, $f\in C([0,1]\times[0,+\infty)\times(-\infty,0],[0,+\infty))$.

This paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we state and prove our main results on the existence and multiplicity of positive solutions for (1.1). At the same time, the corresponding examples are also included to demonstrate the results we obtained.

2 Preliminaries

For convenience, we first state some definitions and preliminary results which we need. Throughout this paper, we make the following assumptions:

(H1) $f \in C([0,1] \times [0,+\infty) \times (-\infty,0], [0,+\infty))$ is continuous;

(H2) $h(t) \in C([0,1])$, $h(t) \le 0$ for all $t \in [0,\xi]$, $h(t) \ge 0$ for all $t \in [\xi, 1]$, where $0 < \frac{1}{4} < \xi < \frac{2}{3} < 1$; and $h(t) \not\equiv 0$ for any subinterval of $[0,1]$.

Denote

$$\overline{f_0} = \limsup_{|u|+|v|\to 0^+} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \quad \overline{f_\infty} = \limsup_{|u|+|v|\to \infty} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \tag{2.1}$$

$$\underline{f_0} = \liminf_{|u|+|v|\to 0^+} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \quad \underline{f_\infty} = \liminf_{|u|+|v|\to \infty} \max_{t\in[0,1]} \frac{f(t,u,v)}{|u|+|v|}, \tag{2.2}$$

and

$$A=\int_\xi^1 G_2(s,s)h(s)\mathrm{d}s,\ B=\min\Big\{\frac{a}{\Delta}\Big(\frac{1}{4}+\frac{3\xi}{4}\Big)\Big(\frac{c}{2}+d\Big)\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} h(s)\mathrm{d}s,\ \frac{a}{2\Delta}\Big(\frac{c}{4}+d\Big)\int_{\frac{1}{4}+\frac{3\xi}{4}}^{\frac{3}{4}} h(s)\mathrm{d}s\Big\},$$