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EQUIVALENCE BETWEEN NONNEGATIVE SOLUTIONS TO PARTIAL SPARSE AND WEIGHTED l_1-NORM MINIMIZATIONS[*][†]

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Abstract

Based on the range space property (RSP), the equivalent conditions between nonnegative solutions to the partial sparse and the corresponding weighted l_1-norm minimization problem are studied in this paper. Different from other conditions based on the spark property, the mutual coherence, the null space property (NSP) and the restricted isometry property (RIP), the RSP-based conditions are easier to be verified. Moreover, the proposed conditions guarantee not only the strong equivalence, but also the equivalence between the two problems. First, according to the foundation of the strict complementarity theorem of linear programming, a sufficient and necessary condition, satisfying the RSP of the sensing matrix and the full column rank property of the corresponding sub-matrix, is presented for the unique nonnegative solution to the weighted l_1-norm minimization problem. Then, based on this condition, the equivalence conditions between the two problems are proposed. Finally, this paper shows that the matrix with the RSP of order k can guarantee the strong equivalence of the two problems.

Keywords compressed sensing; sparse optimization; range space property; equivalent condition; l_0-norm minimization; weighted l_1-norm minimization

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1 Introduction

In this paper, we consider the following partial sparse minimization problem

$$\min_{x,y} \sum_{i=1}^{n_1} w_i |x_i|_0 + a^{\mathrm{T}} y \quad s.t.\ Ax + By = b,\ x \geq 0,\ y \geq 0, \tag{1}$$

where $x = (x_1, x_2, \cdots, x_{n_1})^{\mathrm{T}} \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$. $|x_i|_0 = 0$ if $x_i = 0$; otherwise $|x_i|_0 = 1$.

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$w_i \in R$ is the weight on $|x_i|_0$, $i = 1, 2, \cdots, n_1$. $a \in \mathbb{R}^{n_2}$, $A \in \mathbb{R}^{m\times n_1}$, $B \in \mathbb{R}^{m\times n_2}$, and $b \in \mathbb{R}^m$ $(m < n_1 + n_2)$ are the problem datas. Let $\|x\|_0$ be the number of nonzero components of x, that is, $\|x\|_0 = \sum_{i=1}^{n_1} |x_i|_0$. Although $\|x\|_0$ is not a norm, we still call it l_0-norm for simplicity.

By relaxing $|x_i|_0$ as $|x_i|$, and taking into account $x \geq 0$, $\sum_{i=1}^{n_1} w_i|x_i| = w^{\mathrm{T}}x$, we get the following linear program

$$\min_{x,y} \; w^{\mathrm{T}}x + a^{\mathrm{T}}y \quad s.t.\; Ax + By = b,\; (x, y) \geq 0, \tag{2}$$

where $w = (w_1, w_2, \cdots, w_{n_1})^{\mathrm{T}}$. We are interested in what conditions can ensure the equivalence of problems (1) and (2).

In recent years, l_0-norm minimization problems have been widely researched, and have been successfully applied to signal processing [1], pattern recognition [2], machine learning [3], computational biology [4], medical imaging [5], and other fields [6-9]. Recent research indicates that l_1-norm relaxation can promote sparsity [11]. This is based on equivalence between the l_0-norm and l_1-norm minimization problems.

Up to now, the study of the equivalence between l_0-norm and l_1-norm minimization problems is mainly for the following two problems:

$$\min_{x} \; \|x\|_0 \quad s.t.\; Ax = b, \tag{3}$$

and

$$\min_{x} \; \|x\|_1 \quad s.t.\; Ax = b. \tag{4}$$

It has been proved that, all k-sparse solutions to problem (3) can be found by solving problem (4), if the spark of the sensing matrix A is greater than $2k$ [19], or the order of the null space property (NSP) [21] or the restricted isometry property (RIP) of A is $2k$ or above [25]. However, these conditions remain restrictive and are hard to be verified. From geometric perspective, Donoho and Tanner [30] showed that the outward k-neighborliness of A could also guarantee the equivalence of problems (3) and (4). Donoho and Romberg [11] also analysed the equivalence from probabilistic perspective.

Based on the RSP, Zhao [26,27] presented equivalent conditions for problems (3) and (4), no matter whether the nonnegative constraints exist or not. The conditions guarantee not only the strong equivalence but also the equivalence between the l_0-norm and l_1-norm minimization problems. Moreover, Zhao presented an RSP which could be verified easily, by solving a linear programming problem [26].

In this paper, we consider the equivalence between problems (1) and (2). The definition of equivalence is similar to that in [26], which is as follows.