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WELL-POSEDNESS AND SPACE-TIME REGULARITY OF SOLUTIONS TO THE LIQUID CRYSTAL EQUATIONS IN CRITICAL SPACE*†

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Abstract

In this paper, we consider a hydrodynamic flow of nematic liquid crystal system. We prove the local well-posedness for the system in the critical Lebesgue space, and study the space-time regularity of the local solution.

Keywords space-time regularity; liquid crystal system; critical Sobolev space

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1 Introduction

In this paper, we consider the following hydrodynamic flow of nematic liquid crystal system:

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d), \quad \text{in } \mathbb{R}^n \times (0, +\infty), \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \quad \text{in } \mathbb{R}^n \times (0, +\infty), \\ (u(x,t), d(x,t))|_{t=0} = (u_0(x), d_0(x)), \quad |d_0(x)| = 1, \end{cases} \tag{1.1}$$

which was proposed by Lin and Liu [25,26], as a simplified system of Ericksen-Leslie model. Here u is the velocity of the flow, $d(\cdot, t) : \mathbb{R}^n \to \mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$, is the unit vector field to depict the macroscopic molecular orientation of nematic liquid crystal material, P is pressure. We denote by $\nabla d \otimes \nabla d$ the 3×3-matrix whose (i, j)-entry is $\nabla_i d \cdot \nabla_j d$ and $1 \leq i, j \leq 3$.

The hydrodynamic theory of liquid crystal flow due to Ericksen and Leslie was developed in 1960's [5,6,21,22]. The model (1.1) is a simplified system of Ericksen-Leslie model, and it is a macroscopic continuum description of the time evolution of

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material under the influence of both the flow field $u(x,t)$ and the macroscopic description of the microscopic orientation configuration $d(x,t)$ of rod-like liquid crystal.

Many efforts on rigorous mathematical analysis of system (1.1) have been made, see [23,25-27,29] etc. Since the liquid crystal system (1.1) is a coupling system between the incompressible Navier-Stokes equations and the heat flow of harmonic maps, we shall first recall some results of Navier-Stokes equations as follows.

For the incompressible Navier-Stokes equations, in [19], Leray proved that for any finite square-integrable initial data there exists a (possibly not unique) global-in-time weak solution. Moreover, for two space dimensions case, [20] proved the uniqueness of the weak solution. Although the problems of uniqueness and regularity for $n \geq 3$ of Leray-Hopf weak solutions are still open, since the seminal work of Leray, there is an extensive literature on conditional results under various criteria. The most well-known condition is so-called Ladyzhenskaya-Prodi-Serrin condition, that is for some $T > 0$, $u \in L^p(0,T;L^q(\mathbb{R}^n))$, where the pair (p,q) satisfies

$$\frac{2}{p} + \frac{n}{q} \leq 1, \quad q \in (n, +\infty]. \tag{1.2}$$

Under condition (1.2), the uniqueness of Leray-Hopf weak solutions was proved by Prodi [33] and Serrin [34], and the smoothness was obtained by Ladyzhenskaya [15]. The borderline case $(p,q) = (\infty, n)$ is much more subtle.

Subsequently, [8] proved the well-posedness for the Navier-Stokes equations in a scaling invariant space $\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$. The scaling invariant in the context of the Navier-Stokes equations is defined as: if a pair of functions $(u(x,t), P(x,t))$ solves the incompressible Navier-Stokes equations, then

$$(u_\lambda, P_\lambda)(x,t) = (\lambda u(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t)) \tag{1.3}$$

is also the solution of the incompressible Navier-Stokes equations with initial data $(u_\lambda(x,0), P_\lambda(x,0)) = (\lambda u_0(\lambda x), \lambda^2 P_0(\lambda x))$. The spaces which are invariant under such a scaling are also called critical spaces. Examples of critical spaces for the Navier-Stokes in n dimensions are:

$$\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n) \subset L^n(\mathbb{R}^n) \subset \dot{B}^{-1+\frac{n}{p},p}_{p|p<\infty}(\mathbb{R}^n) \subset BMO^{-1}(\mathbb{R}^n) \subset B^{-1,\infty}_{\infty}(\mathbb{R}^n). \tag{1.4}$$

The study of the Navier-Stokes equations in critical spaces was initiated by Fujita-Kato [8,13], and continued by many authors, see [1,7,10,14,32] etc.

In 2003, Escauriaza, Seregin, and Sverak [7] obtained many perfect results, such as the backward uniqueness of the parabolic system and the regularity results for weak Leary-Hopf solutions u satisfying the additional condition $u \in \mathbb{L}^\infty(0,T;\mathbb{L}^3(\mathbb{R}^3))$, as well as the local well-posedness in the critical Lebesgue space, which verified the borderline case of (1.2) for $n = 3$. The results of [7] is the borderline case for the