Ann. of Appl. Math.
32:2(2016), 133-140

ITERATIVE POSITIVE SOLUTIONS FOR SINGULAR RIEMANN-STIELTJES INTEGRAL BOUNDARY VALUE PROBLEM*†

Xiuli Lin‡, Zengqin Zhao

(*School of Mathematical Sciences, Qufu Normal University, Shandong 273165, PR China*)

Abstract

By applying iterative technique, we obtain the existence of positive solutions for a singular Riemann-Stieltjes integral boundary value problem in the case that $f(t,u)$ is non-increasing respect to u.

Keywords Riemann-Stieltjes integral boundary value problems; positive solution; non-increasing; iterative technique

2000 Mathematics Subject Classification 26A42

1 Introduction

Problems with boundary conditions, especially Riemann-Stieltjes integral boundary condition, have been studied in many papers (see [1-8] and the references therein). In [3], by applying monotone iterative technique, Mao and Zhao established a sufficient condition for the existence of positive solutions for problem (1.1) :

$$\begin{cases} -u''(t) + k^2 u = f(t,u), \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(t) \mathrm{d}A(t), \end{cases} \tag{1.1}$$

where A is right continuous on $[0,1)$, left continuous at $t=1$, and nondecreasing on $[0,1)$, with $A(0)=0$. $\int_0^1 u(t)\mathrm{d}A(t)$ denotes the Riemann-Stieltjes integral of u with respect to A. k is a constant and $f(t,u)$ is increasing with respect to u.

*This work was supported by Program for Scientific research innovation team in Colleges and universities of Shandong Province, the Doctoral Program Foundation of Education Ministry of China (20133705110003), the Natural Science Foundation of Shandong Province of China (ZR2014AM007), the National Natural Science Foundation of China (11571197).

†Manuscript received October 6, 2015; Revised March 17, 2016

‡Corresponding author. E-mail: lin-xiuli78@163.com

In this paper, we consider the case that $f(t,u)$ is non-increasing with respect to u, and $f(t,u)$ may be singular at $u=0,\ t=0$ (and/or $t=1$). By searching an iterative initial element, we construct a non-monotonic iterative sequence which has non-decreasing and non-increasing subsequence to obtain the existence and uniqueness of positive solutions in some set Q. Meanwhile, we also give an error estimate.

2 Preliminaries

The following conditions are assumed in this paper:

(S_1) $f:(0,1)\times(0,+\infty)\longrightarrow[0,+\infty)$ is continuous.

(S_2) For $(t,u)\in(0,1)\times(0,+\infty)$, f is non-increasing respect to u and there exists a constant $\lambda\in(0,1)$ such that for $\tau\in(0,1]$,

$$f(t,\tau u)\le\tau^{-\lambda}f(t,u). \tag{2.1}$$

From (2.1), it is easy to see that if $\tau\in[1,+\infty)$, then

$$f(t,\tau u)\ge\tau^{-\lambda}f(t,u). \tag{2.2}$$

(S_3) There exists a $k>0$ such that $\sinh(k)>\int_0^1\sinh(k(1-t))\mathrm{d}A(t)$.

Lemma 2.1[1] *Assume that $h\in C(0,1)$ and (S_3) holds. Then the following linear boundary value problem*

$$\begin{cases}-u''(t)+k^2u=h(t), & t\in(0,1),\\ u(0)=0, \quad u(1)=\displaystyle\int_0^1 u(t)\mathrm{d}A(t)\end{cases} \tag{2.3}$$

has a unique positive solution u expressed in the following form

$$u(t)=\int_0^1 F(t,s)h(s)\mathrm{d}s,$$

where

$$F(t,s)=G(t,s)+\frac{\sinh(kt)}{\sinh(kt)-\int_0^1\sinh(k\tau)\mathrm{d}A(\tau)}\int_0^1 G(\tau,s)\mathrm{d}A(\tau),\quad t\in[0,1], \tag{2.4}$$

$$G(t,s)=\begin{cases}\dfrac{\sinh(ks)\sinh(k(1-t))}{k\sinh(k)}, & 0\le s\le t\le 1,\\[2mm] \dfrac{\sinh(kt)\sinh(k(1-s))}{k\sinh(k)}, & 0\le t\le s\le 1.\end{cases}$$

Remark 2.1 Assume that (S_1), (S_2) and (S_3) hold. Then solutions for (1.1) are equivalent to continuous solutions of the integral equation

$$u(t)=\int_0^1 F(t,s)f(s,u(s))\mathrm{d}s,$$

where $F(t,s)$ is defined by (2.4).