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THE APPLICATION OF RANDOM MATRICES IN MATHEMATICAL PHYSICS*

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Abstract

In this paper, we introduce the application of random matrices in mathematical physics including Riemann-Hilbert problem, nuclear physics, big data, image processing, compressed sensing and so on. We start with the Riemann-Hilbert problem and state the relation between the probability distribution of nontrivial zeros and the eigenvalues of the random matrices. Through the random matrices theory, we derive the distribution of Neutron width and probability density between energy levels. In addition, the application of random matrices in quantum chromo dynamics and two dimensional Einstein gravity equations is also present in this paper.

Keywords random matrices; Riemann Hypothesis; Riemann-Hilbert problem; nuclear physics

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1 Introduction

A random matrix, that is, every element in the matrix is a random variable, it was first reported in mathematical statistics in 1930, but it didn't get people's attention. The statistical asymptotic properties of its eigenvalues is seldom understood. It was not until 1950 that physicists discovered that the statistical properties of slow neutron resonances were related to random matrices in nuclear physics. In 1951, Wigner [1] pointed out that the local statistical properties of nuclear levels were related to the eigenvalues of random matrices. After that, it is closely related to the quantum chromodynamics, the two-dimensional quantum Einstein gravity equation, the electronic heat capacity of conventional superconducting nano-particles, the

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magnetic susceptibility of conventional superconducting metal nano-particles and superconductivity. In 1962, mathematical physicist F.J. Dyson [2] pointed out an important conjecture: The two point correlation function for the random matrix and the two point correlation function for the zero point of Riemann Zeta function are equivalent. In 1973, H.L. Montgomery gave the mathematical proof of this. Computational mathematician Monte Caro performed a large number of calculations to verify that the distribution of a large number of zeros of $\zeta(s)$ is consistent with the Riemann Hypothesis.

As we known, Riemann proposed the famous Riemann Hypothesis in 1858: The Riemann Zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}\, s > 1,$$

where p labels primes and then by analytic continuation to the rest of the complex plane. It has a single simple pole at $s = 1$, zeros at $s = -2, -4, -6, \cdots$, and infinitely many zeros, called the non-trivial zeros, in $0 < \mathrm{Re}\, s < 1$. The Riemann Hypothesis states that all of the nontrivial zeros lie on the critical line of $\mathrm{Re}\, s = 1/2$. At the Second World Congress of Mathematicians, Hilbert presented twenty-three mathematical problems, one of which was Riemann Hypothesis. To this day, the proof of Riemann Hypothesis has become the most difficult and powerful yet unsolved problem at present, because many important mathematical results could be established follows from the proof of Riemann Hypothesis. Therefore, the random matrix becomes an important tool in proving the Riemann Hypothesis, the computation are used to verify that around 10^9 zeros of ζ function are all on $\mathrm{Re}\, s = 1/2$. In addition, the theory of random matrices also plays an important role in the numerical computation of large data and the robustness of perceptual compression.

2 Orthogonal Polynomial and Riemann-Hilbert Problem

Definition 2.1 Suppose that $w(x)$ is a nonnegative and integrable function on (a, b), where (a, b) is unbounded, then we ask the moments

$$\mu_n = \int_a^b x^n w(x) \mathrm{d}x, \quad n = 0, 1, 2, \cdots \tag{2.1}$$

are finite. If there exists a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ such that

$$\int_a^b P_m(x) P_n(x) w(x) \mathrm{d}x = K_n \delta_{mn}, \quad K_n \neq 0, \tag{2.2}$$

then $\{P_n(x)\}$ is called an orthogonal polynomial sequence with respect to the weight function $w(x)$ on (a, b).