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# SOLUTION FOR TWO-POINT BOUNDARY VALUE PROBLEM OF THE SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATION*†

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**Abstract**

In this paper, we establish the existence result of solution and positive solution for two-point boundary value problem of a semilinear fractional differential equation by using the Leray-Schauder fixed-point theorem. The discussion is based on the system of integral equations on a bounded region.

**Keywords** boundary value problem; Green's function; Leray-Schauder fixed point theorem; system of integral equations

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## 1 Introduction

Fractional differential equations have received increasing attention during the past decades. It has attracted a lot of attention of researchers to promote the continuous development of methods, theories and applications in the field of small area estimation (see [1-3]). Fractional derivative is divided into two categories: standard Riemann-Liouville derivative and Caputo fractional derivative.

The aim of this paper is to study the existence result of solution and positive solution for the following two-point boundary value problem of the semilinear fractional differential equation

$$\begin{cases} D^\alpha u(t) + f(t, u(t), D^{\alpha-1}u(t)) = 0, \quad 0 \leqslant t \leqslant 1, \\ u(0) = 0, \quad u(1) = B, \quad D^{\alpha-1}u(0) = C, \end{cases} \tag{1.1}$$

where $2 < \alpha \leqslant 3$ and $A, B, C$ are real numbers, $D^\alpha$ is the standard Riemann-Liouville derivative, and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous on its domain. Such a

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nonlinearity term $f(t, u(t), D^{\alpha-1}u(t))$ has been studied widely in [6,7]. In [6], by means of the Schauder fixed point theorem and the Banach contraction principle the authors investigated the existence and uniqueness of solutions for a class of nonlinear multi-point boundary value problems for fractional differential equations

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), D^{\beta}u(t)) = 0, \quad 0 \leqslant t \leqslant 1, \\ u(0) = 0, \quad D^{\beta}u(1) - \sum\limits_{i=1}^{m-2} \xi_i D^{\beta}u(\xi_i) = u_0. \end{cases}$$

In [7], by means of a fixed point theorem on a cone, the authors investigated the existence of positive solutions for the following singular fractional boundary value problem

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), D^{\mu}u(t)) = 0, \quad 0 \leqslant t \leqslant 1, \\ u(0) = u(1) = 0. \end{cases}$$

The difference between [6] and [7], the system of integral equations is adopted skillfully in this paper. In the literature of [8], $A = 0$ is the special case of this paper.

## 2 Preliminaries

For convenience, we present here the necessary definitions and some lemmas from fractional calculus theory.

**Definition 2.1**[4] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s$$

provided the right side is pointwise defined on $(0, \infty)$.

**Definition 2.2**[4] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number $\alpha$, provided the right side integral is pointwise defined on $[0, 1)$.

**Lemma 2.1**[4] *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

*has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$, $C_i \in \mathbb{R}$, $i = 1, 2, \cdots, N$, which is a unique solution, where $N$ is the smallest integer greater than or equal to $\alpha$.*

**Lemma 2.2**[4] *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*