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LOCALIZED PATTERNS OF THE SWIFT-HOHENBERG EQUATION WITH A DISSIPATIVE TERM*†

Yongli Liu, Yancong Xu‡

(*Dept. of Math., Hangzhou Normal University, Hangzhou 310036, Zhejiang, PR China*)

Abstract

In this paper, the normal form analysis of quadratic-cubic Swift-Hohenberg equation with a dissipative term is investigated by using the multiple-scale method. In addition, we obtain Hamiltonian-Hopf bifurcations of two equilibria and homoclinic snaking bifurcations of one-peak and two-peak homoclinic solutions by numerical simulations.

Keywords bifurcation; normal form; Swift-Hohenberg equation; Hamiltonian-Hopf; snakes

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1 Introduction

Spatially localized patterns are associated with particular stationary solutions of mathematical models described by partial differential equations, such as the Swift-Hohenberg equation and Ginzburg-Landau equation. These patterns have also been observed in many bistable systems and investigated intensively [1-22], particularly the localised roll patterns, which correspond to homoclinic orbits of the associated ordinary differential equations. For example, homoclinic snaking curve, see Figure 1 for details, has been observed in many reversible hamiltonian systems, here homoclinic snaking refers to a branch curve of homoclinic orbits near a heteroclinic cycle with the increasing width of localized rolls.

Consider the following Swift-Hohenberg equation

$$u_t = -\mu u - (1+\partial_x^2)^2 u + bu^2 - u^3, \quad x \in R. \tag{1}$$

It is easy to see that if we treat μ as a bifurcation parameter, then at $\mu = 0$, the bifurcation is subcritical if $b^2 > \frac{27}{38}$ and supercritical if $b^2 < \frac{27}{38}$. Furthermore, equa-

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‡Corresponding author. E-mail: yancongx@163.com

tion (1) with $u_t = 0$ is a reversible and conservative system, which has the first integral

$$H_1(u) = \frac{1}{2}(\mu+1)u^2 + u_x^2 - \frac{1}{2}u_{xx}^2 + u_x u_{xxx} + \frac{1}{3}bu^3 - \frac{1}{4}u^4. \tag{2}$$

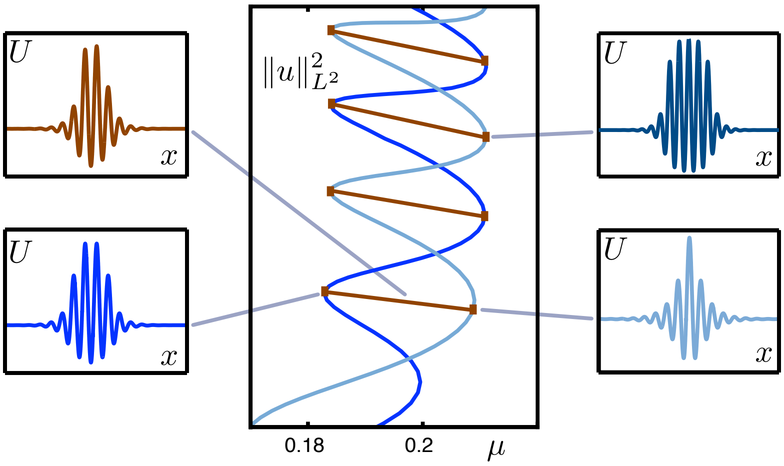

Figure 1: Homoclinic snaking curve

Figure 1 shows snaking bifurcation of symmetric solutions, laddering bifurcation of non-symmetric solutions and four sample solution profiles of Swift-Hohenberg equation (1). Also the brown solution profile is non-symmetric, each of the baby blue solution profiles is symmetric with a maximum and the blue solution profile is symmetric with a minimum.

Burke et al. [2] studied a modified Swift-Hohenberg equation

$$u_t = -\mu u - (1+\partial_x^2)^2 u + bu^2 - u^3 + \gamma u_{xxx} \tag{3}$$

with $b = 2$, which is a perturbation of (1) with a dissipative term γu_{xxx}, which destroy both the reversibility symmetry and variational property. As suggested by the authors that the snakes and ladders structure had been broken into a stack of isolas rather than snakes. Knobloch et al. [10] proved the existence of the isolas of 2-pulse solutions about stationary 1D patterns of the normal quadratic-cubic Swift-Hohenberg equation.

In this paper, we consider a Swift-Hohenberg equation with a non-reversible and conservative term, which also presents the existence of one-pulse and two-pulse snakes with a stack of saddle-nodes rather than a stack of pitch-forks. More precisely, we consider the normal form and bifurcation of the following perturbed Swift-Hohenberg equation with a dissipative term

$$u_t = -\mu u - (1+\partial_x^2)^2 u + bu^2 - u^3 + \alpha u_x u_{xx}, \tag{4}$$

which is a variational and non-reversible system when $\alpha \neq 0$. We use both μ and b