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# LOCALIZED PATTERNS OF THE CUBIC-QUINTIC SWIFT-HOHENBERG EQUATIONS WITH TWO SYMMETRY-BREAKING TERMS*†

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**Abstract**

Homoclinic snake always refers to the branches of homoclinic orbits near a heteroclinic cycle connecting a hyperbolic or non-hyperbolic equilibrium and a periodic orbit in a reversible variational system. In this paper, the normal form of a Swift-Hohenberg equation with two different symmetry-breaking terms (non-reversible term and non-$k$-symmetry term) are investigated by using multiple scale method, and their bifurcation diagrams are initially studied by numerical simulations. Typically, we predict numerically the existence of so-called round-snakes and round-isolas upon particular two symmetric-breaking perturbations.

**Keywords** round-snakes; round-isolas; normal form; Swift-Hohenberg equation; localized patterns

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## 1 Introduction

In recent years, there has been a lot of interest in localized patterns in some partial differential equations on the real line [4-24], including numerous physical, biological and chemical models [1,3]; see also [2,11] for additional references. As we know, these localized states can bifurcate from the trivial state. At the same time, they often exhibit homoclinic snaking when they approach a spatially periodic structure [18,19], especially in some reversible and variational systems [5,7,11,21]. Note that, the term snaking refers to the back-and-forth oscillations of the localized patterns within the parameter space region as localized states grow in width. As a

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matter of fact, the term "spatially localized pattern" refers to particular stationary solution of the partial differential equations. The addressed localized roll patterns correspond to homoclinic orbits of the associated ordinary differential equations, and the equilibrium and periodic orbit corresponds to the background state and regular periodic pattern of partial differential equations, respectively. As a specified example, we consider the general cubic-quintic Swift-Hohenberg equation

$$u_t = ru - (1 + \partial_x^2)^2 u + bu^3 - u^5, \quad x \in \mathbb{R}, \tag{1.1}$$

with $b = 2$, which is invariant under the symmetry $R : x \to -x$, $k : u \to -u$. It has been studied by a number of authors [5,21]. This equation has two kinds of symmetries: $R : x \to -x$ (reversible) and $k : u \to -u$, the effect of $k \circ R$ is to force the drift speed to vanish. In fact, the asymmetric states are also equilibria due to the variational structure of the cubic-quintic Swift-Hohenberg equation. Note that, these asymmetric states are known to be unstable. It is possible to make the odd-parity states drift by breaking the midplane reflection symmetry $k$ of the system.

As we know, this equation has the usual snakes and ladders, and there are three kinds of patterns: a snaking branch representing even-parity solutions, that is, solutions invariant with respect to $R$, a snaking branch representing odd-parity solutions with respect to $k \circ R$, and the rung states. The rung states connect the two snaking branches, arising in pitch-fork bifurcations close to the saddle-node bifurcations on the snaking branches. Moving up along the snake branch, the saddle nodes converge exponentially rapidly to a fixed values of the parameter $r$ and do so from the same side at both boundaries of the snaking region. At the same time, the pitchfork bifurcations leading to the rung states converge exponentially rapidly to the saddle-node bifurcations.

Stationary localized patterns of (1.1) can be reduced to homoclinic orbits of the steady-state equation

$$0 = ru - (1 + \partial_x^2)^2 u + bu^3 - u^5, \quad x \in \mathbb{R}. \tag{1.2}$$

This equation can be written as the first-order ordinary differential equation

$$u_x = f(u, \mu), \quad u = (u_1, u_2, u_3, u_4) = (u, u_x, u_{xx}, u_{xxx}) \in \mathbb{R}^4, \tag{1.3}$$

which is a reversible Hamiltonian system.

The structure of variational

$$F[u(x)] \triangleq \int_{-\infty}^{+\infty} \mathrm{d}x \Big\{ -\frac{1}{2} ru^2 + \frac{1}{2}[(1 + \partial_x^2)u]^2 - \frac{1}{4} bu^4 + \frac{1}{6} u^6 \Big\} \tag{1.4}$$

satisfies $\frac{\partial u}{\partial t} = -\frac{\delta F}{\delta u}$ and generates the following Hamiltonian

$$H_0(u) \triangleq -\frac{1}{2}(r-1)u^2 + [\partial_x u]^2 - \frac{1}{2}[\partial_x^2 u]^2 + \partial_x u \partial_x^3 u - \frac{1}{4} bu^4 + \frac{1}{6} u^6, \tag{1.5}$$