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Boundary Homogenization of a Class of Obstacle Problems

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Abstract. We study the homogenization of a boundary obstacle problem on a $C^{1,\alpha}$-domain D for some elliptic equations with uniformly elliptic coefficient matrices γ. For any $\epsilon\in\mathbb{R}_+$, $\partial D=\Gamma\cup\Sigma$, $\Gamma\cap\Sigma=\emptyset$ and $S_\epsilon\subset\Sigma$ with suitable assumptions, we prove that as ϵ tends to zero, the energy minimizer u^ϵ of $\int_D|\gamma\nabla u|^2dx$, subject to $u\geq\varphi$ on S_ε, up to a subsequence, converges weakly in $H^1(D)$ to $\tilde{u}$, which minimizes the energy functional

$$\int_D|\gamma\nabla u|^2+\int_\Sigma(u-\varphi)^2_-\mu(x)dS_x,$$

where $\mu(x)$ depends on the structure of S_ϵ and φ is any given function in $C^\infty(\overline{D})$.

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1 Introduction

Let $D\subset\mathbb{R}^n$ $(n>2)$ be a bounded open subset, whose boundary satisfies

$$\partial D=\Gamma\cup\Sigma,\quad \Gamma\cap\Sigma=\emptyset \quad \text{and} \quad \partial D\in C^{1,\alpha}, \tag{1.1}$$

for some constant $\alpha\in(0,1)$. For any $\epsilon\in\mathbb{R}_+$, let S_ϵ be a subset of Σ with some special structure, which will be specified later. Throughout, we assume:

(a1) $\gamma(x)=(\gamma_{ij}(x))_{n\times n}$ is an $n\times n$ symmetric matrix-valued function on D and there exist two positive constants a and b with $a\leq b$ such that

$$aI\leq\gamma(x)\leq bI, \quad x\in D,$$

where I is the identity matrix;

(a2) φ and ψ are both smooth functions defined on $\overline{D}$.

Consider the following variational problem

$$\inf_{v\in K} J(v), \tag{1.2}$$

where

$$J(v):=\int_D |\gamma\nabla v|^2 dx, \tag{1.3}$$

and

$$K:=\{v\in H^1(D): v|_\Gamma=\psi \ \text{ and } \ v|_{S_\epsilon}\geq\varphi\}. \tag{1.4}$$

Let u^ϵ be the solution to the variational problem (1.1)–(1.3). Here we focus on the study of the asymptotic behavior of u^ϵ when $\epsilon\to 0$ under suitable assumptions on S_ϵ and Σ. That is, we are concerned with the boundary homogenization associated with the variational problem (1.2). This is an important problem with many practical applications. For instance, (1.2) can be used to describe the mathematical model for semipermeable membranes, where the function $\varphi(x)$ signifies an external pressure, and the set S_ϵ is considered as a subset of the boundary composed of the part through which the liquid passes on the semipermeable membrane. Interested readers may refer to [9] and the references therein for more details.

In case that $\gamma(x)$ is the identity matrix I, there is a long history in studying (1.2) with rich results in the literature. When the set S_ϵ lies inside D, the problem can be viewed as the homogenization of a variational problem on a perforated domain. For this problem, [6, 7] firstly considered the periodic homogenization and established that the limiting energy functional contains a strange term which depends on the