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The Relation Between a Tensor and Its Associated Semi-Symmetric Form

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Abstract. It is known that every tensor has an associated semi-symmetric tensor. The purpose of this paper is to investigate the shared properties of a tensor and its semi-symmetric form. In particular, a corresponding semi-symmetric tensor has smaller Frobenius norm under some conditions and can be used to get smaller bounds for eigenvalues and solutions of dynamical systems and tensor complementarity problems. In addition, every tensor has the same eigenvalues as its corresponding semi-symmetric form, also a corresponding semi-symmetric tensor inherits properties like being circulant, Toeplitz, Z-tensor, M-tensor, H-tensor and some others. Also, there are a two-way connection for properties like being positive definite, P-tensor, semi-positive, primitive and several others.

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1. Introduction

Tensors are generalizations of matrices and vectors. Since tensors have many applications in different sciences [5,12,50] and because of the increased computing capacity of computers, the attention on tensors has been raised in the recent decade. As linear algebra has a solid theory, there have been efforts in the literature to generalize the theoretical and numerical frameworks of linear algebra to tensors [2, 3, 15, 16, 18, 19, 26,

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27,41,47,53,55,56,59]. For example, tensor complementarity problems have been defined and used in applications like game theory, hypergraph clustering problem, traffic equilibrium problems, magnetic resonance imaging and etc. [23,25,58].

A semi-symmetric tensor is a tensor which is symmetric in all indices but the first one, for every tensor there exists a semi-symmetric tensor which has a close relationship with its mother tensor and inherits many properties such as eigenvalues from it. Since semi-symmetric tensors have a simpler structure, this relation can be useful for obtaining smaller bounds for different tensor problems. For example, consider the following dynamical system [14]:

$$\frac{dx^{[m-1]}}{dt} = \mathcal{A}x^{(m-1)},$$

where $\mathcal{A}$ is a tensor, there exists a semi-symmetric tensor $\mathcal{B}$ such that $\mathcal{A}x^{(m-1)} = \mathcal{B}x^{(m-1)}$, using $\mathcal{B}$ in this dynamical system can lead to obtaining a smaller bound for solutions of the system (Section 4.1). There are many tensor problems that $\mathcal{A}x^{(m-1)}$ appears in them, for example tensor complementarity problems, various tensor eigenvalue problems and etc. For tensor complementarity problems, using operator bounds we can have smaller bounds for their solutions (Section 4.2). Because the corresponding semi-symmetric tensor has a simpler structure, in particular it has a smaller frobenius norm under some conditions, using it can lead to having smaller bounds. Also, it preserves many properties from its mother tensor which we study them later (as Theorem 3.1). As examples, we have considered attitude dynamics of a rigid body (Example (3.1)), a randomly-generated dynamical system (Example (4.1)), a multi-Person noncooperative game (Example (4.3)), and compliance tensors (Example (4.7)) and some others.

The rest of the paper is structured as follows. We first give some basic definitions [40,41]. Then, in Section 2, the corresponding semi-symmetric tensors and the ways to obtain them are given. In Section 3, the main properties of corresponding semi-symmetric tensors are studied as Theorem 3.1. In Section 4, we give the applications of Theorem 3.1 for obtaining smaller bounds for eigenvalues and solutions of dynamical systems and tensor complementarity problems. Finally, in Section 5, we discuss the storage differences of a tensor and its associated semi-symmetric tensor. Section 6 presents a summary of the results and conclusions.

We now define the eigenvalues and eigenvectors of real tensors. Thus we need first to define the real tensor product with a vector of complex entries.

Definition 1.1. *Suppose* $\mathcal{A} = a(i_1, i_2, \ldots, i_m)$*,* $a(i_1, i_2, \ldots, i_m) \in \mathbb{R}$*,* $i_1, i_2, \ldots, i_m \in \{1, \ldots, n\}$ *is an* m*-th order tensor and* $x = (x_1, x_2, \ldots, x_n)^t$ *is a vector in* $\mathbb{C}^n$*, then for* $r \in \{1, \ldots, m-1\}$*, the tensor multiplication by a vector is defined as follows:*

$$\left(\mathcal{A}x^{m-r}\right)_{i_1 \ldots i_r} := \sum_{i_{r+1}, \ldots, i_m = 1}^{n} a(i_1, i_2, \ldots, i_m) x_{i_{r+1}} x_{i_{r+2}} \ldots x_{i_m}.$$

Next, we define different tensor eigenvalue problems.