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Additive Inexact Block Triangular Preconditioners for Saddle Point Problems Arising in Meshfree Discretization of Piezoelectric Equations

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Abstract. Additive inexact block triangular preconditioners for discretized two-dimensional piezoelectric equations are proposed. In such preconditioners, the (1,1) leading block is used as the block diagonal part of a discrete elasticity operator. The (2,2) block is the approximation of the exact Schur complement matrix. It is additively assembled by a small exact Schur complement matrix in each background cell. The proposed preconditioners are easy to construct and have sparse structure. It is proved that (1,1) and (2,2) blocks of the preconditioners are spectrally equivalent to the (1,1) block of the discretized piezoelectric equation and the exact Schur complement matrix, respectively. Two numerical examples show that Krylov subspace iteration methods preconditioned in this way, are fast convergent and the iteration steps do not depend on the degree of freedom.

AMS subject classifications: 65F10, 65F50

Key words: Piezoelectric equation, element-free Galerkin method, saddle point problem, block triangular preconditioner, inexact Schur complement matrix.

1. Introduction

Piezoelectric devices play crucial role in medicine, military, automotive sensors, actuators, and micro-electro-mechanical systems [51, 53]. The basic characteristic of piezoelectric devices is the coupling of electrical and mechanical phenomena. More exactly, an electrical voltage is generated if the piezoelectric material is subjected to a mechanical deformation and a strain is created if a voltage is applied to a piezoelectric material. Therefore, accurate and efficient electromechanical analysis is very important in many practical applications of piezoelectric materials.

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Let ∇ denote the gradient operator, A^T be the transpose of a matrix or a vector A, $\nabla\cdot\mathbf{F}$ the divergence of a vector field $\mathbf{F}$, and

$$\nabla_s^T = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

the symmetric gradient operator.

We consider a piezoelectric body occupying a two-dimensional domain Ω in the $x-z$ Cartesian coordinate system and bounded by a surface Γ with the outward unit normal vector $\mathbf{n}$. To model, simulate and analyze the coupled electromechanical behavior of the piezoelectric structures for static problem, we have to solve the following system of equations:

$$\left.\begin{aligned} &\nabla_s^T \boldsymbol{\sigma} + \mathbf{f_b} = 0, \\ &\boldsymbol{\varepsilon} = \nabla_s \mathbf{u}, \\ &\nabla\cdot\mathbf{D} = 0, \\ &\mathbf{E} = -\nabla\phi, \\ &\begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{c} & -\mathbf{e}^T \\ \mathbf{e} & \mathbf{k}^{\varepsilon} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \mathbf{E} \end{bmatrix}, \end{aligned}\right. \quad \text{in } \Omega \tag{1.1}$$

with the natural mechanical and electrical boundary conditions

$$\begin{aligned} \boldsymbol{\sigma}\mathbf{n} &= \bar{\mathbf{t}} && \text{on } \Gamma_\sigma, \\ \mathbf{D}\mathbf{n} &= -\bar{\mathbf{q}} && \text{on } \Gamma_q, \end{aligned} \tag{1.2}$$

and the essential mechanical and electric boundary conditions

$$\begin{aligned} \mathbf{u} &= \bar{\mathbf{u}} && \text{on } \Gamma_u, \\ \phi &= \bar{\phi} && \text{on } \Gamma_\phi. \end{aligned} \tag{1.3}$$

The first equation in (1.1) connecting the stress $\boldsymbol{\sigma}$ and the body force $\mathbf{f_b}$, is called the equilibrium equation. The second equation in (1.1), where $\boldsymbol{\varepsilon}$ denotes the strain and $\mathbf{u}$ the displacement, is called the geometry equation. The third equation in (1.1) containing the electric displacement $\mathbf{D}$, is called the electrostatic equilibrium equation. The fourth equation in (1.1) describes the relation between the electric field $\mathbf{E}$ and the electric potential ϕ. The fifth equation in (1.1) is called the generalized constitutive equation. Here, $\mathbf{c}\in\mathbb{R}^{3\times 3}$, $\mathbf{e}\in\mathbb{R}^{2\times 3}$ and $\mathbf{k}^{\varepsilon}\in\mathbb{R}^{2\times 2}$ are the elasticity matrix measurement at constant electric field, the piezoelectric matrix and the dielectric matrix at constant mechanical strain, respectively. A bar over a quantity stands for the prescribed values.

In general, it is very hard to solve the Eqs. (1.1). Nevertheless, this system can be reduced to the following coupled mechanical-electrical partial differential equations:

$$\left.\begin{aligned} &\nabla_s^T(\mathbf{c}\nabla_s\mathbf{u}) + \nabla_s^T\big(\mathbf{e}^T\nabla\phi\big) + \mathbf{f_b} = 0, \\ &\nabla\cdot(\mathbf{e}\nabla_s\mathbf{u}) - \nabla\cdot(\mathbf{k}^{\varepsilon}\nabla\phi) = 0, \end{aligned}\right. \quad \text{in } \Omega \tag{1.4}$$