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On Relaxed Greedy Randomized Augmented Kaczmarz Methods for Solving Large Sparse Inconsistent Linear Systems

Zhong-Zhi Bai[1,2], Lu Wang[1,2,*] and Galina V. Muratova[3]

[1] *State Key Laboratory of Scientific/Engineering Computing, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100190, P.R. China.*
[2] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, P.R. China.*
[3] *Laboratory of Computational Mechanics, I.I. Vorovich Institute of Mathematics, Mechanics and Computer Science, Southern Federal University, Rostov-on-Don 344090, Russia.*

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Abstract. For solving large-scale sparse inconsistent linear systems by iteration methods, we introduce a relaxation parameter in the probability criterion of the greedy randomized augmented Kaczmarz method, obtaining a class of relaxed greedy randomized augmented Kaczmarz methods. We prove the convergence of these methods and estimate upper bounds for their convergence rates. Theoretical analysis and numerical experiments show that these methods can perform better than the greedy randomized augmented Kaczmarz method if the relaxation parameter is chosen appropriately.

AMS subject classifications: 65F10, 65F20, 65K05, 90C25, 15A06

Key words: System of linear equations, relaxation, augmented linear system, randomized Kaczmarz method, convergence property.

1. Introduction

We consider iterative solution of the large-scale sparse linear system

$$Ax = b, \quad \text{with} \quad A \in \mathbb{C}^{m\times n} \quad \text{and} \quad b \in \mathbb{C}^{m}, \tag{1.1}$$

that is, A is a complex $m\times n$ matrix, b is a complex m-dimensional vector, and x is a complex n-dimensional unknown vector. We also introduce an augmented linear system associated

[*] Corresponding author. *Email addresses:* bzz@lsec.cc.ac.cn (Z.-Z. Bai), wanglu@lsec.cc.ac.cn (L. Wang), muratova@sfedu.ru (G.V. Muratova)

with the above linear system (1.1), which is

$$\begin{pmatrix} I_m & A \\ A^* & 0 \end{pmatrix}\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \tag{1.2}$$

where A, x, and b are defined as before, I_m is an $m\times m$ identity matrix, and z is a complex m-dimensional auxiliary vector. In addition, A^* denotes the conjugate transpose of the matrix $A\in\mathbb{C}^{m\times n}$.

The Kaczmarz method, also known as the algebraic reconstruction technique (ART), is a classical and effective iteration method for solving the linear system (1.1). In 2009, Strohmer and Vershynin [14] proposed the randomized Kaczmarz (RK) method by sweeping through the rows randomly rather than sequentially, and proved its expected exponential rate of convergence. Afterwards, many computationally effective iteration methods have been proposed and implemented on the basis of the RK method, in order to further improve its convergence rate and enlarge its application range. For instance, Zouzias and Freris [16] introduced the randomized extended Kaczmarz (REK) method to handle the problem that the RK method can not converge when the linear system (1.1) is inconsistent; Needell and Tropp [13] discussed a block randomized Kaczmarz method to accelerate the convergence rate of the RK method; Bai and Wu [4, 5] initiated the probability criterion of the row selection to grasp larger entries of the residual vector at each iteration step, proposing the greedy randomized Kaczmarz (GRK) method, which converges significantly faster than the RK method; and, more recently, Bai and Wu [8] further proposed the greedy randomized augmented Kaczmarz (GRAK) method, which technically applies the GRK method to the equivalent consistent augmented linear system (1.2). The GRAK method can be widely used in solving large sparse system of linear equations especially in the inconsistent case, and it can be much more effective than the REK method in numerical experiments. We refer to [2, 3, 6, 7, 10–12, 15] for more details on the analysis and implementations of the RK method and its generalized variants.

In this paper, we introduce a relaxation parameter in the probability criterion of the GRAK method, obtaining a so-called relaxed greedy randomized augmented Kaczmarz (RGRAK) method for solving the large-scale, sparse, and inconsistent linear system $Ax = b$ in (1.1). In other words, we can derive the RGRAK method by directly applying the relaxed greedy randomized Kaczmarz (RGRK) method given in [5] to the equivalent consistent augmented linear system (1.2). We prove the convergence of the RGRAK method, and give an upper bound for its convergence rate. Moreover, in theory we demonstrate that this upper bound is smaller than that of the GRAK method when the relaxation parameter $\theta \geq 1/2$. Numerical experiments also confirm that the RGRAK method outperforms the GRAK method in terms of both iteration steps and computing times, provided we have an appropriate choice of the relaxation parameter θ.

The organization of this paper is as follows. In Section 2 we describe the RGRAK method and analyze its convergence property. In Section 3 we report numerical results. We end this paper with conclusions and remarks in Section 4.