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A Block Fast Regularized Hermitian Splitting Preconditioner for Two-Dimensional Discretized Almost Isotropic Spatial Fractional Diffusion Equations

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Abstract. Block fast regularized Hermitian splitting preconditioners for matrices arising in approximate solution of two-dimensional almost-isotropic spatial fractional diffusion equations are constructed. The matrices under consideration can be represented as the sum of two terms, each of which is a nonnegative diagonal matrix multiplied by a block Toeplitz matrix having a special structure. We prove that excluding a small number of outliers, the eigenvalues of the preconditioned matrix are located in a complex disk of radius $r < 1$ and centered at the point $z_0 = 1$. Numerical experiments show that such structured preconditioners can significantly improve computational efficiency of the Krylov subspace iteration methods such as the generalized minimal residual and bi-conjugate gradient stabilized methods. Moreover, if the corresponding equation is almost isotropic, the methods constructed outperform many other existing preconditioners.

AMS subject classifications: 65F08, 65F10, 65N06, 65N22, CR:G1.3

Key words: Preconditioning, spatial fractional diffusion equation, Toeplitz matrix, two-dimensional problem.

1. Introduction

Fractional differential equations often appear in the modelling of various anomalous diffusion phenomena arising in turbulent flows [13, 32], image processing [1], and other applications [12, 17, 20, 27, 29, 34]. However, analytical solutions of such equations are

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rarely available. In this work, we deal with the numerical solution of the two-dimensional almost-isotropic spatial fractional diffusion equations

$$\begin{aligned} & -\omega(x,y)\left[\frac{\partial^{\beta_1}u(x,y)}{\partial_+x^{\beta_1}}+\frac{\partial^{\beta_2}u(x,y)}{\partial_+y^{\beta_2}}\right] \\ & \quad -\gamma(x,y)\left[\frac{\partial^{\beta_1}u(x,y)}{\partial_-x^{\beta_1}}+\frac{\partial^{\beta_2}u(x,y)}{\partial_-y^{\beta_2}}\right]=f(x,y), \quad (x,y)\in D, \\ & u(x,y)=0, \qquad (x,y)\in\partial D, \end{aligned} \tag{1.1}$$

where $\omega(x,y)$, $\gamma(x,y)$ are the diffusion coefficients such that

$$\begin{aligned} & \omega(x,y)\approx\gamma(x,y), \\ & \omega(x,y)>0, \ \ \gamma(x,y)>0, \\ & (x,y)\in D=(0,1)\times(0,1), \end{aligned}$$

$f(x,y)$ is the source term, and $\frac{\partial^{\beta_1}u(x,y)}{\partial_+x^{\beta_1}}$, $\frac{\partial^{\beta_2}u(x,y)}{\partial_+y^{\beta_2}}$ and $\frac{\partial^{\beta_1}u(x,y)}{\partial_-x^{\beta_1}}$, $\frac{\partial^{\beta_2}u(x,y)}{\partial_-y^{\beta_2}}$ are, respectively, the left and right Riemann-Liouville fractional derivatives of order $\beta_1,\beta_2\in(1,2)$ [26,30].

Using the shifted finite-difference scheme of Grünwald-Letnikov type [22,37], we can obtain a discrete linear system for the two-dimensional spatial fractional diffusion equation (1.1). This system has a full, dense, and ill-conditioned coefficient matrix. Therefore, the computational cost of the direct methods based on linear solvers such as the Gaussian elimination [10,15] has the complexity $\mathcal{O}(n^3)$. In addition, the storage $\mathcal{O}(n^2)$, where n is the product of inner grid-points of variables x and y, is needed — [6,21,23]. On the other hand, since the corresponding coefficient matrix has a special structure, one can employ the fast Fourier transform (FFT) in the matrix-vector multiplication and obtain a method, which requires $\mathcal{O}(n)$ storage and has $\mathcal{O}(n\log n)$ computational complexity — cf. [36,37].

Such approach with different type of preconditioners and Krylov subspace iteration methods — e.g. the generalized minimal residual (GMRES) and bi-conjugate gradient stabilized (BiCGSTAB) methods [3, 10], is an important tool in solving spatial fractional diffusion equations. In particular, if the diffusion coefficients $\omega(x,y)$ and $\gamma(x,y)$ are sufficiently smooth or nearly linear, the circulant-based approximate inverse (CAI) preconditioner from [25] can be also successfully used in two-dimensional cases. For other recent results on this topic the reader can consult [18, 24] and references therein. Donateli *et al.* [14] proposed a tridiagonal approximation (TA) preconditioner and a block tridiagonal approximation (BTA) preconditioner for one- and two-dimensional equations, respectively. However, if the fractional orders are not close to integers 1 and 2 or if the equation is high-dimensional, the corresponding LU-factorization of the block tridiagonal preconditioner requires a lot of storage. If one diffusion coefficient is much larger than the other, fast appropriately scaled Hermitian and skew-Hermitian splitting (FRSHSS) preconditioners are developed in [4], and further modified in [19].

If the diffusion coefficients are almost equal — i.e. if the spatial fractional diffusion equation (1.1) is almost isotropic, Bai and Lu [8] constructed regularized Hermitian splitting (RHS) iteration methods and proved their asymptotic convergence. Replacing the