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Fast Second-Order Evaluation for Variable-Order Caputo Fractional Derivative with Applications to Fractional Sub-Diffusion Equations

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Abstract. In this paper, we propose a fast second-order approximation to the variable-order (VO) Caputo fractional derivative, which is developed based on $L2\text{-}1_\sigma$ formula and the exponential-sum-approximation technique. The fast evaluation method can achieve the second-order accuracy and further reduce the computational cost and the acting memory for the VO Caputo fractional derivative. This fast algorithm is applied to construct a relevant fast temporal second-order and spatial fourth-order scheme ($FL2\text{-}1_\sigma$ scheme) for the multi-dimensional VO time-fractional sub-diffusion equations. Theoretically, $FL2\text{-}1_\sigma$ scheme is proved to fulfill the similar properties of the coefficients as those of the well-studied $L2\text{-}1_\sigma$ scheme. Therefore, $FL2\text{-}1_\sigma$ scheme is strictly proved to be unconditionally stable and convergent. A sharp decrease in the computational cost and the acting memory is shown in the numerical examples to demonstrate the efficiency of the proposed method.

AMS subject classifications: 35R11, 65M06, 65M12

Key words: Variable-order Caputo fractional derivative, exponential-sum-approximation method, fast algorithm, time-fractional sub-diffusion equation, stability and convergence.

1. Introduction

Over the last few decades, the fractional calculus has gained much attention of both mathematics and physical science due to the non-locality of fractional operators. In fact the fractional calculus has been widely applied in various fields including the biology, the ecology, the diffusion, and the control system [3, 15, 19, 21, 27, 29–31]. Recently, more and more researchers revealed that many important dynamical problems exhibit the fractional order behavior that may vary with time, space, or some other factors, which leads to the concept of the variable-order (VO) fractional operator, see [13, 22,

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38,40]. This fact indicates that the VO fractional calculus is an expected tool to provide an effective mathematical framework to characterize the complex problems in fields of science and engineering; for instances, anomalous diffusion [7,8,39,45], viscoelastic mechanics [9,10,16,35], and petroleum engineering [28].

The VO fractional derivative can be regarded as an extension of the constant-order (CO) fractional derivative [41]. An interesting extension of the classical fractional calculus was proposed by Samko and Ross [33] in which they generalized the Riemann-Liouville and Marchaud fractional operators in the VO case. Later, Lorenzo and Hartley [22] first introduced the concept of VO operators. According to the concept, the order of the operator is allowed to vary as a function of independent variables such as time and space. Afterwards, various VO differential operators with specific meanings were defined. Coimbra [9] gave a novel definition for the VO differential operator by taking the Laplace-transform of the Caputo's definition of the fractional derivative. Soon, Coimbra, and Kobayashi [36] showed that Coimbra's definition was better suited for modeling physical problems due to its meaningful physical interpretations, see also [32]. Moreover, the Coimbra's VO fractional derivative could be considered as the Caputo-type definition, which is defined as follows [9,38]:

$$ {}_0^C\mathcal{D}_t^{\alpha(t)}u(t) \equiv \frac{1}{\Gamma\big(1-\alpha(t)\big)}\int_0^t \frac{u'(\tau)}{(t-\tau)^{\alpha(t)}}\mathrm{d}\tau, \tag{1.1} $$

where $\Gamma(\cdot)$ denotes the Gamma function and $\alpha(t) \in [\underline{\alpha}, \overline{\alpha}] \subset (0,1)$ is the VO function. We remark that, in statistical physics community, the operator (1.1) has clear physical meaning if α is a function of the space variable x (see [13]), while the physical significance for the case of $\alpha(t)$ is still unclear. Nonetheless, the operator (1.1) has been widely used to model some phenomena in engineering community [9,38].

Since the problems described by the VO fractional operator are difficult to handle analytically, possible numerical implementations of the VO fractional problems are given. In [44], Zhao *et al.* derived two second-order approximations for the VO Caputo fractional derivative and provided the error analysis. For the VO time-fractional sub-diffusion equations, Du *et al.* [11] proposed $L2$-1_σ scheme, which makes use of the piecewise high-order polynomial interpolation of the solution. The resulting method was investigated to be unconditionally stable and second-order convergent via the energy method.

In consequence of the nonlocal property and historical dependence of the fractional operators, the aforementioned numerical methods always require all previous function values, which leads to an average $\mathcal{O}(n)$ storage and computational cost $\mathcal{O}(n^2)$, where n is the total number of the time levels. To overcome this difficulty, many efforts have been made to speed up the evaluation of the CO fractional derivative [2,4,14,17,18, 23–25,42]. Nevertheless, the coefficient matrices of the numerical schemes for the VO fractional problems lose the Toeplitz-like structure and the VO fractional derivative is no longer a convolution operator. Therefore, those fast methods for the CO fractional derivative cannot be directly applied to VO cases. Recently, Fang *et al.* [12] proposed a fast algorithm for the VO Caputo fractional derivative based on a shifted binary block