Numer. Math. Theor. Meth. Appl. Vol. **15**, No. 1, pp. 91-124
doi: 10.4208/nmtma.OA-2021-0141 February 2022

On Discontinuous and Continuous Approximations to Second-Kind Volterra Integral Equations

Hui Liang*

School of Science, Harbin Institute of Technology, Shenzhen, 518055, China

Received 13 August 2021; Accepted (in revised version) 16 September 2021

Abstract. Collocation and Galerkin methods in the discontinuous and globally continuous piecewise polynomial spaces, in short, denoted as DC, CC, DG and CG methods respectively, are employed to solve second-kind Volterra integral equations (VIEs). It is proved that the quadrature DG and CG (QDG and QCG) methods obtained from the DG and CG methods by approximating the inner products by suitable numerical quadrature formulas, are equivalent to the DC and CC methods, respectively. In addition, the fully discretised DG and CG (FDG and FCG) methods are equivalent to the corresponding fully discretised DC and CC (FDC and FCC) methods. The convergence theories are established for DG and CG methods, and their semi-discretised (QDG and QCG) and fully discretized (FDG and FCG) versions. In particular, it is proved that the CG method for second-kind VIEs possesses a similar convergence to the DG method for first-kind VIEs. Numerical examples illustrate the theoretical results.

AMS subject classifications: 45D05, 65R20

Key words: Volterra integral equations, collocation methods, Galerkin methods, discontinuous Galerkin methods, convergence analysis.

1. Introduction

In this paper, we consider the following Volterra integral equations (VIEs) of the second kind:

$$u(t) = g(t) + \int_0^t K(t,s)u(s)\,ds, \quad t \in I := [0,T] \tag{1.1}$$

with continuous kernel, which arise widely as mathematical models of physical and biological phenomena (see the two monographs [1,2]). There are lots of researchers focus on the numerical methods of VIEs (see [1,6–9,11–19,21] and the references cited

*Corresponding author. *Email addresses:* `lianghui@hit.edu.cn`; `wise2peak@126.com` (H. Liang)

therein). Among these numerical methods, the collocation method is one of the most popular methods for (1.1). In general, collocation solutions are sought either in the space

$$S_{m-1}^{(-1)}(I_h) := \{v : v|_{\sigma_n} \in \mathcal{P}_{m-1}(\sigma_n)\ (0 \le n \le N-1)\}$$

of discontinuous piecewise polynomials of degree $m-1 \ge 0$, or in the space of globally continuous piecewise polynomials of degree $m \ge 1$ (see [1,13,14])

$$S_m^{(0)}(I_h) := \{v \in C(I) : v|_{\sigma_n} \in \mathcal{P}_m(\sigma_n)\ (0 \le n \le N-1)\}.$$

Here,

$$I_h := \{t_n := nh : n = 0, \ldots, N\ (t_N := T)\}$$

denotes a given mesh on $I = [0,T]$, with mesh diameter $h := T/N$, and $\mathcal{P}_k = \mathcal{P}_k(\sigma_n)$ is the linear space of (real) polynomials on $\sigma_n := [t_n, t_{n+1}]$ of degree not exceeding k. For a given mesh I_h the set

$$X_h := \{t_{n,i} := t_n + c_i h :\ 0 < c_1 < \cdots < c_m \le 1\ (0 \le n \le N-1)\}$$

will denote the collocation points corresponding to prescribed collocation parameters $\{c_i\}$.

In [13], it is proved that the collocation solution in the globally continuous piecewise polynomial space $S_m^{(0)}(I_h)$ for the second-kind VIE (1.1) converges to the exact solution, if and only if the collocation parameters $\{c_i\}$ satisfy the following condition:

$$-1 \le l_0(1) = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \le 1, \tag{1.2}$$

where $l_0(s)$ is defined in (3.3) of Section 3.1.1. It is very interesting that (1.2) is also the sufficient and necessary condition for the collocation solution in the discontinuous polynomial space $S_{m-1}^{(-1)}(I_h)$ converging to the exact solution of the following first-kind VIE (see [1, Theorem 2.4.2]):

$$0 = g(t) + \int_0^t K(t,s)u(s)\,ds, \quad t \in I := [0,T] \tag{1.3}$$

with $|K(t,t)| \ge k_0 > 0$. It means that the continuous collocation (CC) method in the globally continuous piecewise polynomial space $S_m^{(0)}(I_h)$ for the second-kind VIE (1.1) has a similar convergence to the discontinuous collocation (DC) method in the discontinuous piecewise polynomial space $S_{m-1}^{(-1)}(I_h)$ for the first-kind VIE (1.3).

In 2009, [3] investigated the discontinuous Galerkin (DG) method in the discontinuous piecewise polynomial space $S_{m-1}^{(-1)}(I_h)$ for the first-kind VIE (1.3), and it is proved that an $(m-1)$-th degree DG approximation exhibits global convergence of order $m-1$ when m is even and order m when m is odd. A natural question is that does the continuous Galerkin (CG) method in the globally continuous piecewise polynomial space