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A Robust Modified Weak Galerkin Finite Element Method for Reaction-Diffusion Equations

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Abstract. In this paper, a robust modified weak Galerkin (MWG) finite element method for reaction-diffusion equations is proposed and investigated. An advantage of this method is that it can deal with the singularly perturbed reaction-diffusion equations. Another advantage of this method is that it produces fewer degrees of freedom than the traditional WG method by eliminating the element boundaries freedom. It is worth pointing out that, in our method, the test functions space is the same as the finite element space, which is helpful for the error analysis. Optimal-order error estimates are established for the corresponding numerical approximation in various norms. Some numerical results are reported to confirm the theory.

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1. Introduction

The weak Galerkin (WG) finite element method is a numerical technique for solving partial differential equations. Since it has been proposed by Wang [28], the WG method has been applied successfully to the discretization of several classes of partial differential equations and variational inequalities, e.g., second-order elliptic problems [2–5, 8, 9, 11, 16, 19, 29, 31], the Stokes equations [25, 30, 33, 34], the Biharmonic equations [18, 21, 27, 35], the Maxwell equations [22], the Oseen equations [15], the

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Helmholtz equations [20, 23], the multi-term time-fractional diffusion equations [36], the parabolic integro-differential equations [37], the interface problems [17, 24], the natural convection problems [6], and the elliptic variational inequality [26].

It is known that the weak function defined in the WG method has the form $v = \{v_0, v_b\}$ with $v = v_0$ inside of the element and $v = v_b$ on the boundary of the element. The introduction of the weak function and the corresponding weak derivative makes the WG method highly flexible. However, it creates additional degrees of freedom associated v_b. In order to eliminate the unknowns associated with the element boundaries, Wang [31] and Gao [5] modified the WG method for second-order elliptic problems by replacing v_b with the average of v_0 : $\{v_0\}$. It means that the weak function in the modified weak Galerkin (MWG) method has the form $v = \{v_0, \{v_0\}\}$. Therefore, the MWG method has fewer unknowns than the traditional WG method. In [5, 31], the finite element space for second-order elliptic problems is defined by

$$V_h := \big\{v : v|_T \in P_k(T) \ \text{ for } \ T \in \mathcal{T}_h\big\}, \tag{1.1}$$

and the test function space by

$$V_h^0 := \big\{v : v \in V_h, v|_e = 0 \ \text{ for } \ e \in \partial\Omega\big\}.$$

However, the error $e_h = u_h - Q_0 u$ between the MWG solution and the L^2 projection of the exact solution does not necessarily belong to the test function space V_h^0. This will make it difficult to analyse the error.

In this paper, we consider the following diffusion-reaction equations which seeks an unknown function $u = u(\boldsymbol{x})$ satisfying

$$-\nabla \cdot (\mathcal{A}\nabla u) + cu = f \quad \text{in } \ \Omega, \tag{1.2}$$

$$u = 0 \qquad \text{on } \ \partial\Omega, \tag{1.3}$$

where Ω is a polygonal/plolyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$, $\mathcal{A}$ is a symmetric matrix, and $c = c(\boldsymbol{x}) \in L^\infty(\Omega)$ is a scalar positive function on Ω. Assume that the matrix $\mathcal{A}$ satisfies the following property: there exists a constant $\lambda > 0$ such that

$$\xi^t \mathcal{A}\xi \geq \lambda \xi^t \xi, \quad \forall \xi \in \mathbb{R}^d, \tag{1.4}$$

where ξ is understood as a column vector and ξ^t is the transpose of ξ.

Letting $\mathcal{A} = \varepsilon$, where ε is a constant coefficient and $0 < \varepsilon \ll 1$, one can rewrite the problem (1.2)-(1.3) by the following singularly perturbed reaction-diffusion equations: seek an unknown function $u = u(\boldsymbol{x})$ such that

$$-\varepsilon \Delta u + cu = f \quad \text{in } \ \Omega, \tag{1.5}$$

$$u = 0 \qquad \text{on } \ \partial\Omega. \tag{1.6}$$

It is well-known that standard finite element methods often suffer from the deterioration of numerical accuracy for convection-dominated problems. A lot of research has